**Challenge 9: Consecutive Sums**

**Solution**

a) There are 2 ways of getting 100

9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 = 100

18 + 19 + 20 + 21 + 22 = 100

b) There are 5 ways of getting 126

41 + 42 + 43 = 126

30 + 31 + 32 + 33 = 126

15 + 16 + 17 + 18 + 19 + 20 + 21 = 126

10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 = 126

5 + 6 +7 + 8 + 9 + 10 + 11 +12 + 13 + 14 + 15 + 16 = 126

How did they do it?
Each of the three solvers used a different technique. It is a very good exercise to work through their reasoning seeing if you agree with it. If your teacher is stuck for something for you to do in lessons, I suggest you do this!

First student wrote:
I found that whenever you add an ODD number of consecutive integers together, the total is equal to the number in the set multiplied by the number "in the middle".
For example:
111 + 112 + 113 = 336
Which is also (3 x 112)
And whenever you add an EVEN number of consecutive integers together, the total is equal to the number in the set multiplied by the number halfway between the two numbers "closest to the middle".
For example:
111 + 112 + 113 + 114 = 450
Which is also (4 x 112.5)

Second student wrote:
Let a be the number which we want to express as the sum of consecutive integers. Let n, n+1, n+2,…, n+k be some sequence of consecutive integers which when summed give a. Thus:
a = n + n+1 + n+2 +…+ n+k
a = (k+1)(2n+k)/2, as the sum of n consecutive integers b1,b2,…,bn is n(b1+bn)/2
2a =(k+1)(2n+k)
Now consider a), where a=100:
200=(k+1)(2n+k)
K,n are integers ≥1, so we can consider the factors of 200. Note that:
K+1 < 2n+k, because
1 < 2n. Thus we only need to consider when k+1 is smaller than 2n+k

Factor pairs of 200:
1,200 2,100 4,50 5,40 10,20
[editor's note: there is one missing here … if you are reading, stop and fill in the missing pair!]
With checking, 5,40 is the only pair with integer solutions, thus the only way to express it is 18+19+20+21+22.(Note the average of these five numbers is a fifth of 100!)
Now with b), where a=126. By the same method, we get:
$252=(k+1)(2n+k)$
Factor pairs of 252:
1,252 2,126 3,84 4,63 6,42 7,36 9,28 12,21 14,18.
By checking, we find that (3,84), (4,67), (7,46), (9,28) and (12,21) give integer solutions. Thus there are five ways.

Third student wrote:
a) Let a string of k consecutive integers sum to 100, with the smallest being m. We have:
$m+m+1+m+2+m+\ldots+m+(k-1) = 100$

$\rightarrow km + \frac{1}{2}(k[k-1]) = 100$
$\rightarrow k | 100 - \frac{1}{2}(k[k-1])$ (*)

[Editor's note. Some nice notation here. a | b means that a divides b, or in other words that b is a multiple of a]

Now the RHS of (*) is positive, so we have that $100 > \frac{1}{2}(k[k-1])$, which gives by rearranging that $k(k-1) < 200$ and thus $k \leq 14$.

Assume k is odd, then $k|\frac{1}{2}(k[k-1]) \rightarrow k|100$. Thus the only odd value of k which satisfies the conditions is 5.

Assume k is even. Then substituting k=2n into (*) yields $2n|100-\frac{1}{2}(2n[2n-1])$. However $\frac{1}{2}(2n[2n-1])$ is odd when n is odd, so the RHS is now not an integer, so n must be even. However, there will be a higher power of 2 on the LHS of (*) than on the RHS, so k cannot be a multiple of 4. Thus k=5.

[Editor's note: there is a problem with the logic in the "however…" sentence, but it is subtle!]

b) Let a string of k consecutive integers sum to 126. We have, like before that:
$k|126 - \frac{1}{2}(k[k-1])$ (**)

As before, the RHS of (**) is positive, so $126 > \frac{1}{2}(k[k-1])$. Solving the inequality yields $k \leq 16$.

Assume k is odd. Then $k|\frac{1}{2}(k[k-1]) \rightarrow k|126 \rightarrow k = 3,7,9$.

Assume k is even. Then substituting k=2n to (**) yields $2n|126 - \frac{1}{2}(2n[2n-1])$. As before, $\frac{1}{2}(2n[2n-1])$ is odd when n is odd, so the RHS is now not an integer, so n must be even. Putting $n=2x \rightarrow k=4x$ into (**) yields $4x|126 - \frac{1}{2}(4x[4x-1])$. Now 126 leaves remainder 2 when divided by 4, so $2x[4x-1]) = 2 \pmod 4 \rightarrow 8x^2-2x = 2 \pmod 4 \rightarrow x=4y\ 1$. But $4x \leq 16 \rightarrow x \leq 4 \rightarrow x=1$ or $3 \rightarrow k=4$ and 12.
So there are 5 possibilities.

About the general case, if k is odd and n is an even number which k consecutive integers sum to, the k can be all odd factors of n such that $k(k-1) < 2n$.
For even k, I think there are solutions iff $n/k = \frac{1}{2} \pmod k$