Mathematical Logic & Dirichlet's Principle

Session 2

Tutor: Jude Mbroh

March 8, 2025

Albion Online Intellectual Club

Table of contents

Homework Solutions

Homework Problem 1: Friends - Solution

Problem: Demonstrate that in any group of five people, there are at least two people who have an identical number of friends within the group.

Key Observations:

- The possible number of friends a person can have is in the set $\{0, 1, 2, 3, 4\}$.
- If one person has 0 friends, no one in the group can have 4 friends.
- If everyone has a nonzero number of friends, the friend counts are in the set $\{1, 2, 3, 4\}$.

Homework Problem 1: Friends - Solution (Continued)

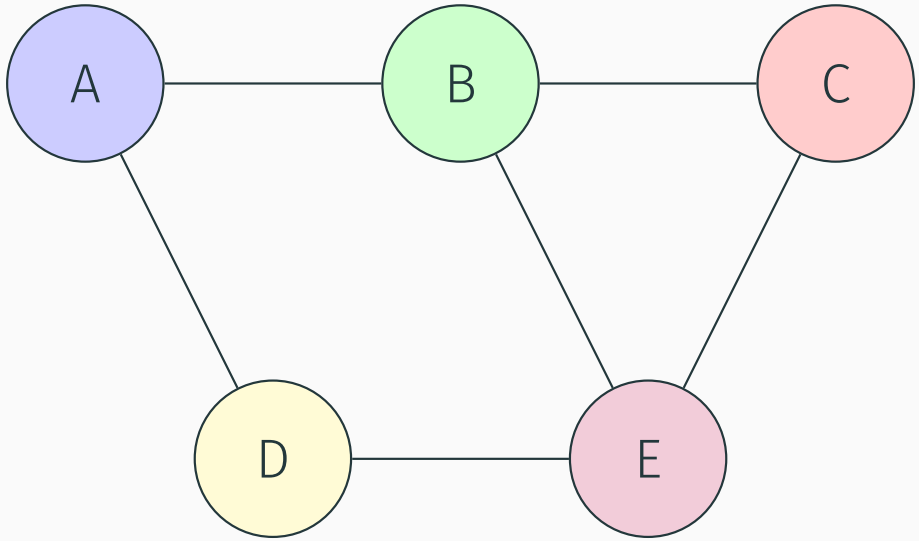
Case 1: Someone has 0 friends.

- The remaining four people have friend counts in the set $\{0, 1, 2, 3\}$.
- If everyone else has a nonzero number of friends, there are 3 options for 4 people.
- By Dirichlet's Principle, at least two people must have the same number of friends.

Case 2: Everyone has at least 1 friend.

- The friend counts are in $\{1, 2, 3, 4\}$.
- There are 4 options for 5 people.
- By Dirichlet's Principle, at least two people must have the same number of friends.

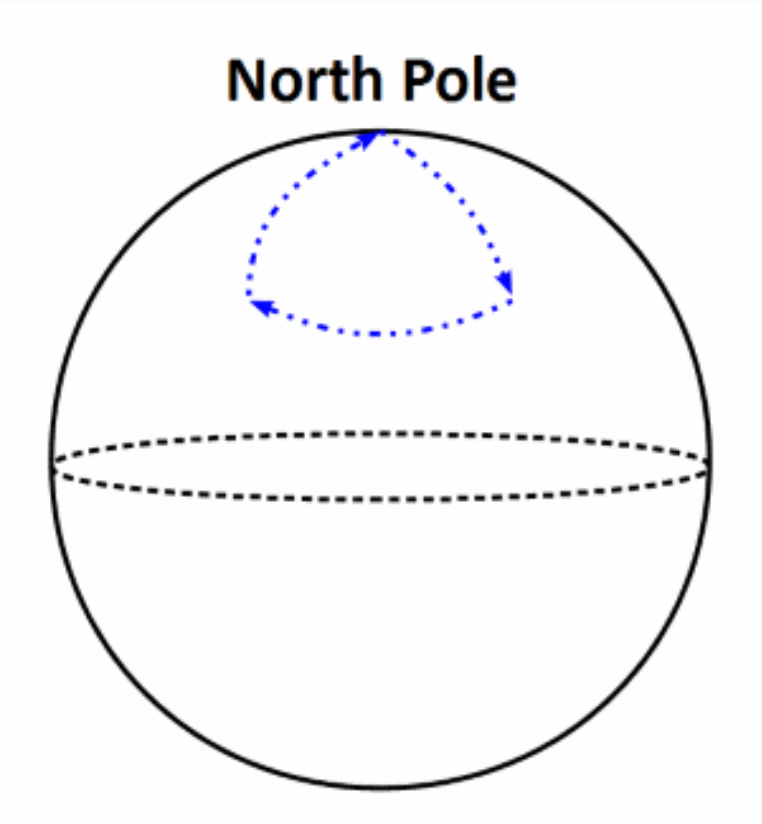
Homework Problem 1 - Solution (Visualisation)



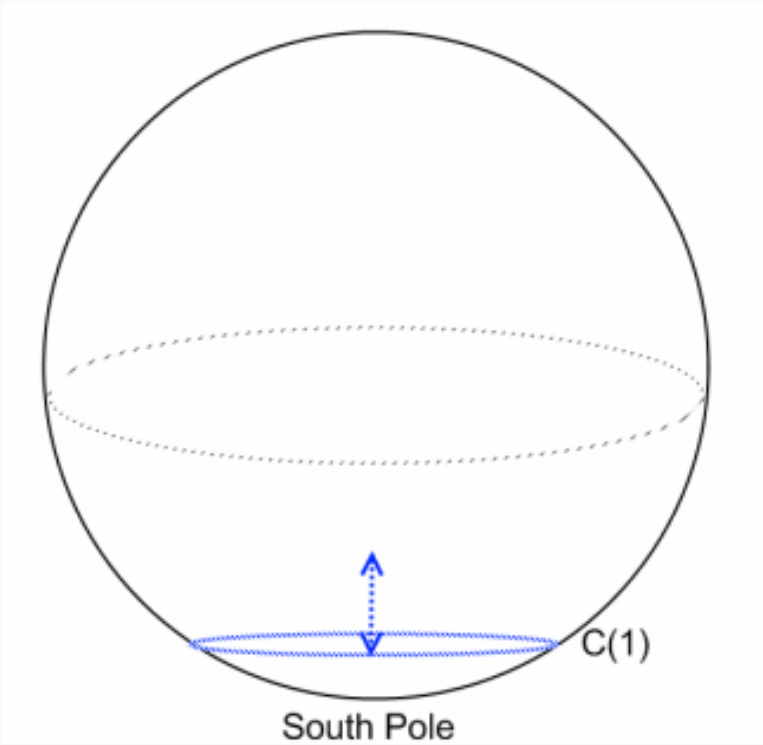
Explanation:

- Each node represents a person.
- Edges represent friendships.
- In this example, two people (B and E) have the same number of friends (3).

Homework Problem 2: The North Pole Journey - Solution



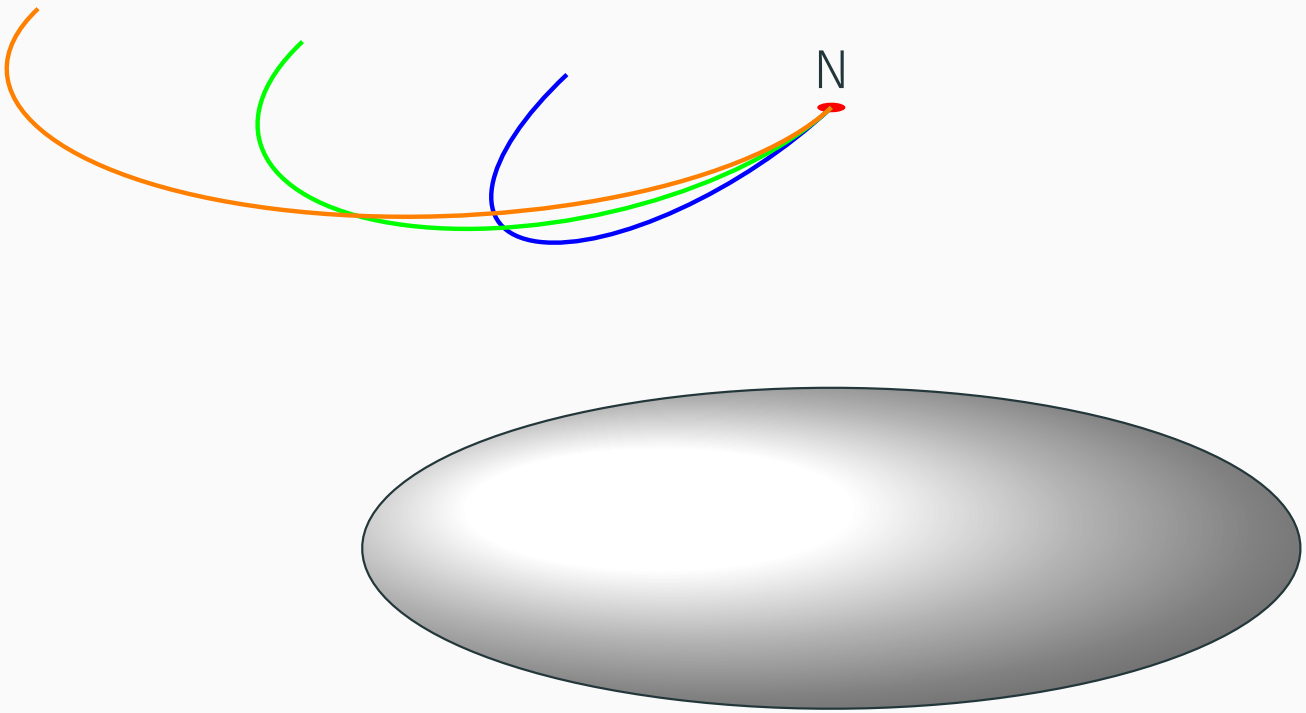
(a) First group of solutions



(b) Second group of solutions

Figure 1: Illustration of all possible solution paths

TikZ 3D Illustration



Homework Problem 3: Prime Numbers - Solution

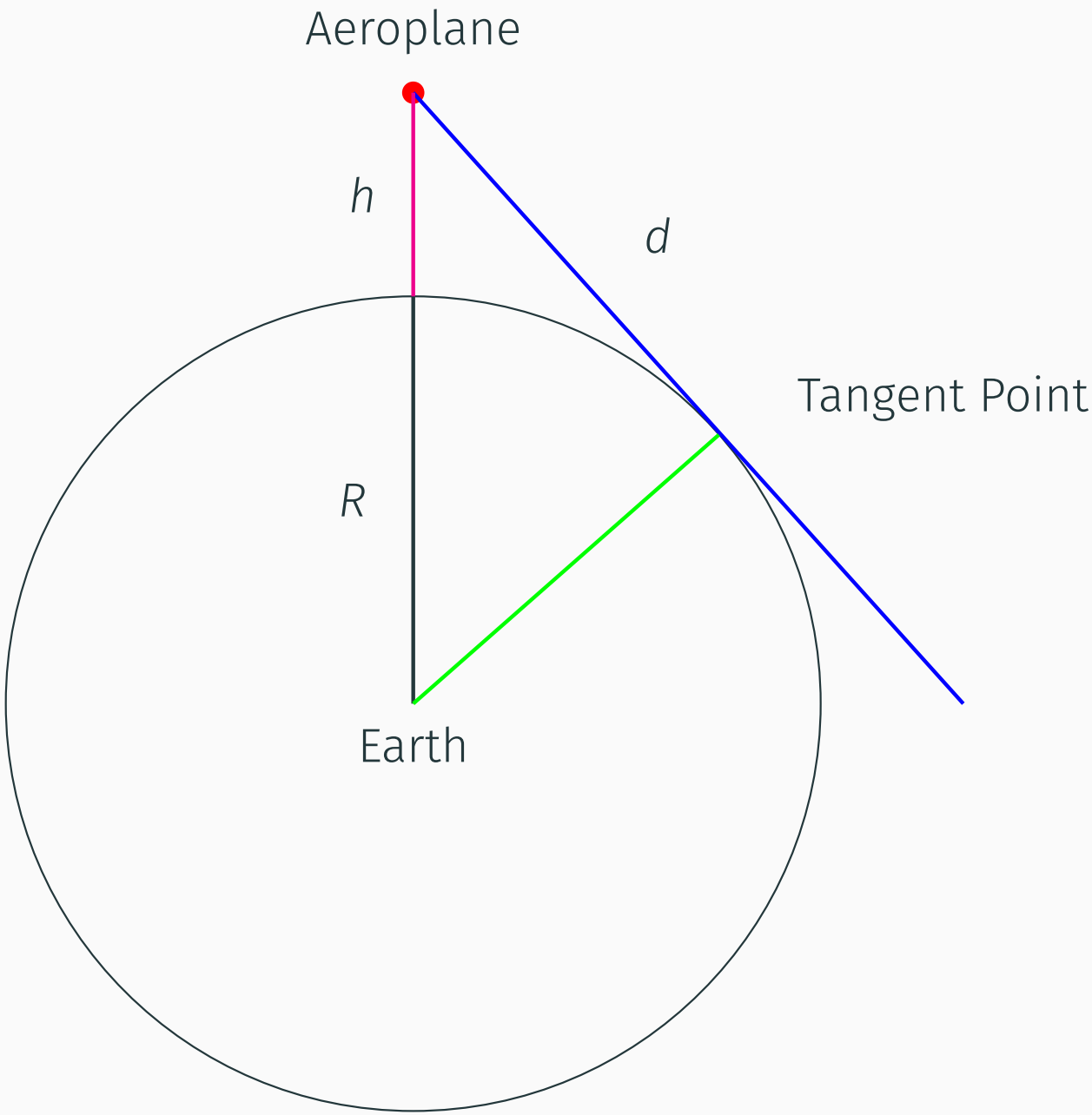
What is a prime number?

A prime number is a number greater than 1 that has no divisors other than 1 and itself. Examples: 2, 3, 5, 7, 11, ...

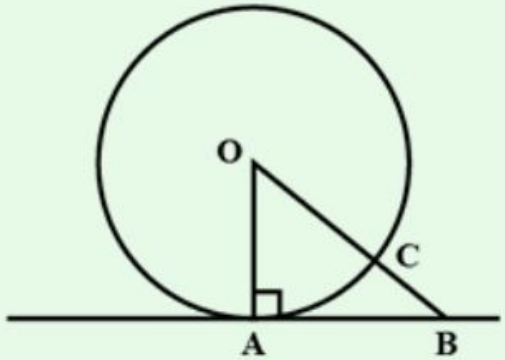
Proof (by the method of contradiction):

1. Assume there are only **finitely many** prime numbers. Let's list them all:
   $p_1, p_2, p_3, \ldots, p_n$.
2. Now, consider the number:
   $N = p_1 \times p_2 \times p_3 \times \cdots \times p_n + 1$.
3. N is either:
   - A **prime number** itself, or
   - Divisible by a prime number **not in our list**.
4. In both cases, we find a prime number **not in our original list**. This contradicts our assumption that we listed all prime numbers.

Homework Problem 4: Distance to the Horizon - Solution



Circle Theorem - The tangent is perpendicular to the radius



Referring to the figure:

$OA = OC$ (Radii of circle)

Now $OB = OC + BC$

$\therefore OB > OC$ (OC being radius and B any point on tangent)

$\Rightarrow OA < OB$

B is an arbitrary point on the tangent.

Thus, OA is shorter than any other line segment joining O to any point on tangent.

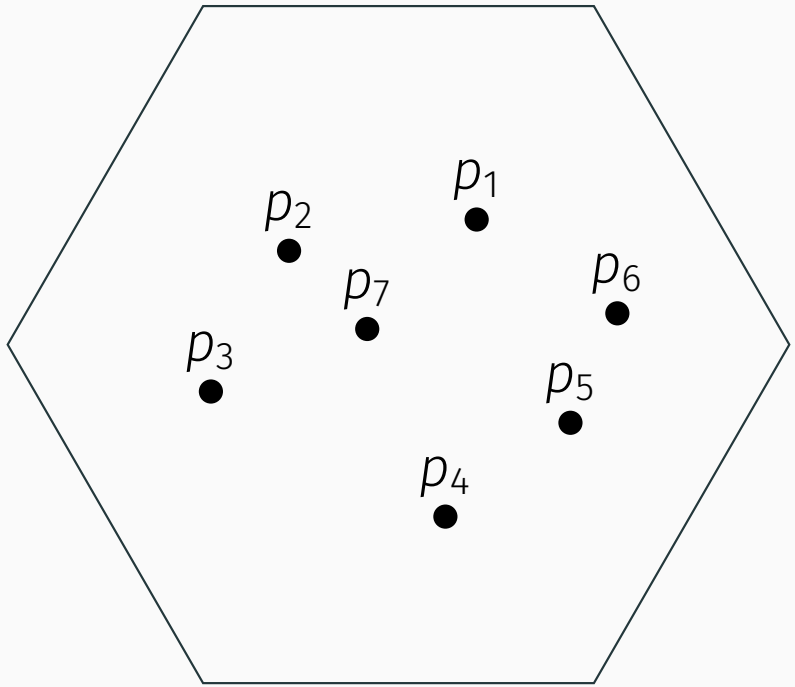
Shortest distance of a point from a given line is the perpendicular distance from that line.

Hence, the tangent at any point of circle is perpendicular to the radius.

Recap

1. Geometry Problem with Dirichlet's Principle

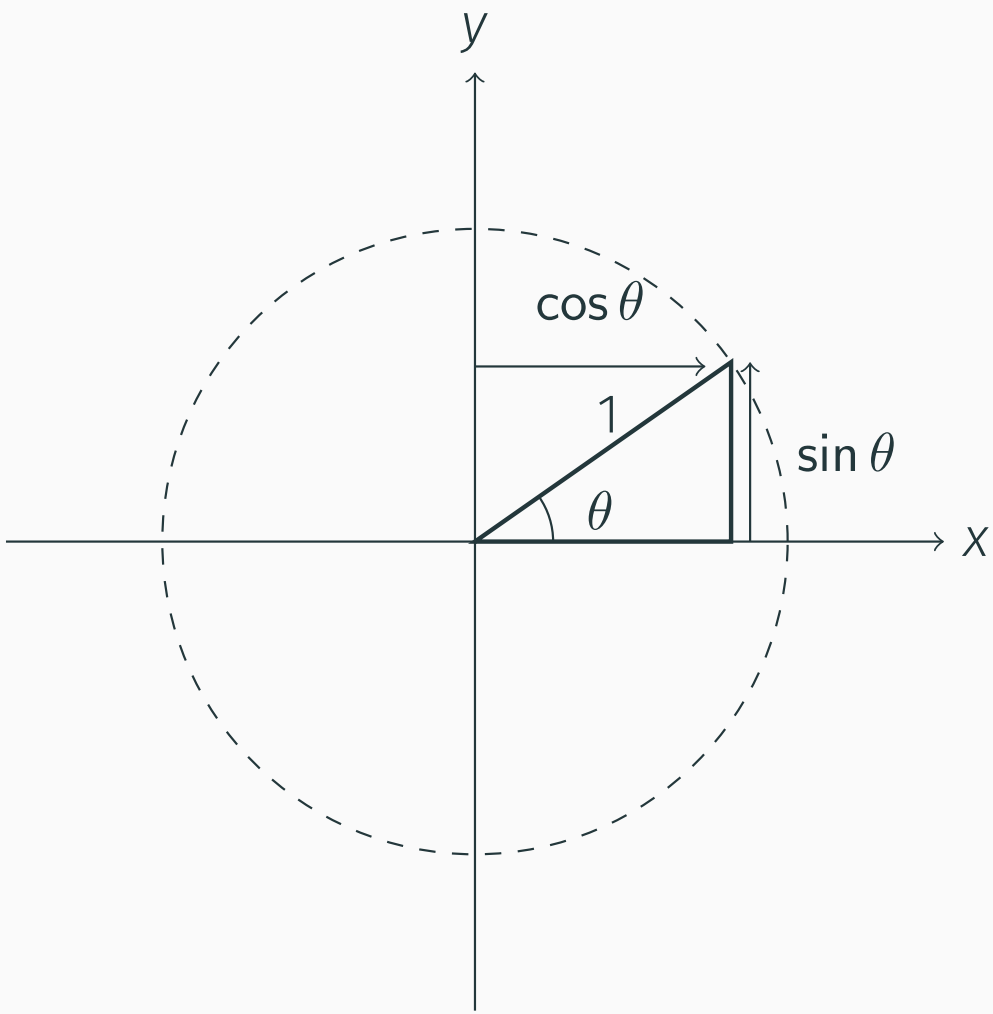
Prove that in any group of 97 points placed inside a regular hexagon of side length 1, at least two points must be at most $\frac{1}{4}$ units apart.



2. Trigonometric Identity Problem

Problem Statement:

Show that $\cos^2\theta + \sin^2\theta \equiv 1$



Mathematical Logic

Task 1 - Coin-Flipping Problem

Problem Statement

- You have n coins, all initially face up, numbered from 1 to n.
- Starting with the second coin, you flip every second coin (i.e., flip coins 2, 4, 6, ...).
- Then, starting from the third coin, you flip every third coin (i.e., flip coins 3, 6, 9, ...).
- Continue this process until you flip the n-th coin.

Question

- After all the flips, which coins will remain facing up?

Task 2 - Bridge Crossing Problem

Problem Statement:

Four people must cross a bridge at night. They have only one flashlight, and the bridge can hold at most two people at a time. The four people have different walking speeds:

- *Person A can cross in 1 minute.*
- *Person B can cross in 2 minutes.*
- *Person C can cross in 5 minutes.*
- *Person D can cross in 10 minutes.*

Question:

When two people cross, they must go at the slower person's pace. What is the fastest total time in which all four people can cross the bridge?

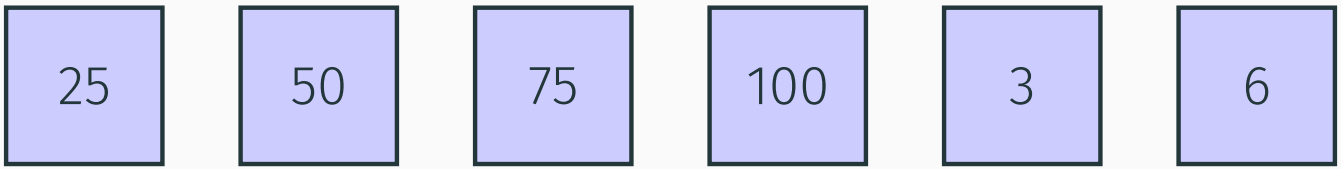
Task 3 - Countdown Numbers Game

Problem Statement

- You are given the numbers: **25, 50, 75, 100, 3, 6**.
- You can use each number **at most once**.
- You can use the operations: **addition (+), subtraction (-), multiplication (×), and division (÷)**.
- Your goal is to reach the **target number: 2,375**.

Question

- How can you combine the numbers and operations to reach the target?

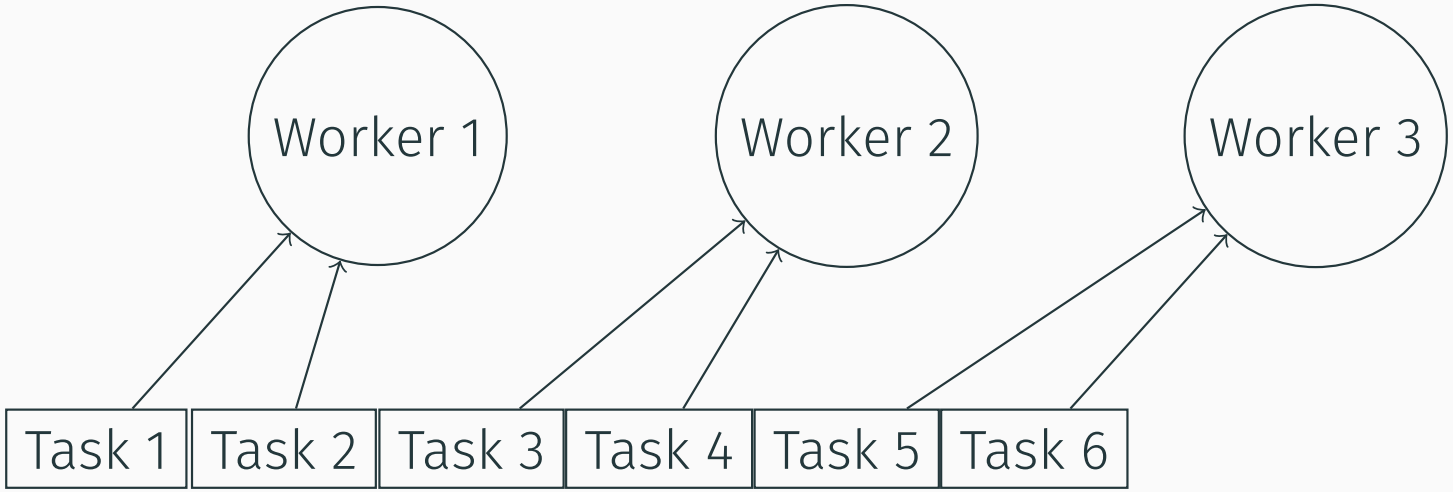


Dirichlet's Principle

Task 4 - Best Work Division

Problem Statement:

Given *n* tasks and *k* workers, what is the most even way to divide the tasks among them?



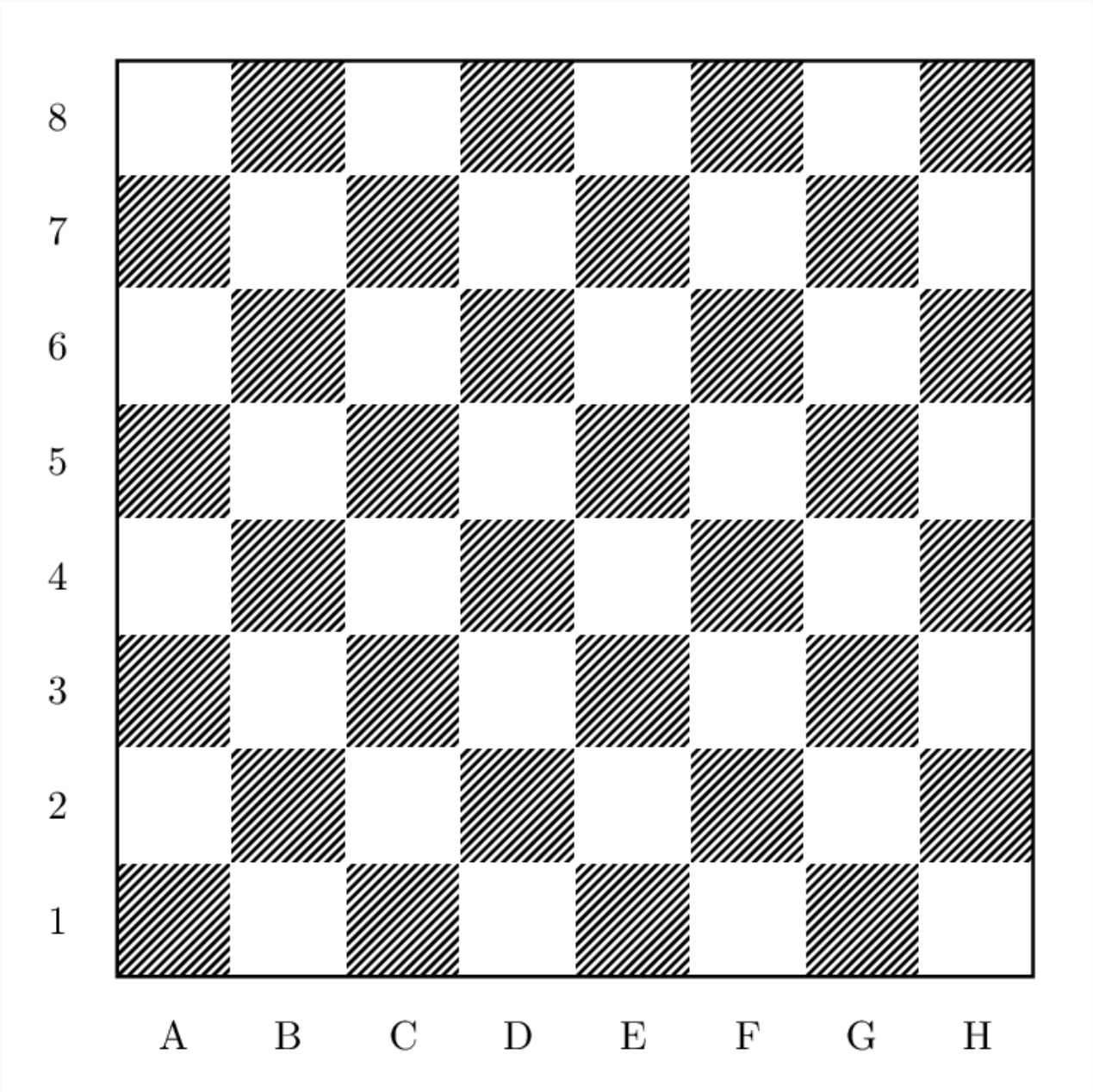
Task 5 - Attacking in Chess

Problem Statement:

In chess, a king can attack any piece in a neighboring square, either vertically, horizontally, and diagonally. How many kings can you put on a 8×8 chessboard so that no king can attack any other?

Question: Can you determine this number for other chess pieces?

Task 5 (Continued) - Visualisation



Conclusion

Questions?