13 Probability

Our futures are uncertain. Will it rain today? Will you enjoy teaching for many years? Will the Cubs win the next World Series? These questions can be answered with words such as "unlikely" or "probably" or with numerical probabilities.

People estimate probabilities using intuition and personal experience, but in some cases, it is possible to give a more precise numerical probability. If a future event is a repetition of (or very similar to) past events, a numerical estimate of its likelihood can be found. Probability is the study of random phenomena that are individually unpredictable but that have a pattern in the long run.

Although archaeologists have found dice that are 5,000 years old, the formal study of probability did not begin until the 16th and 17th centuries in Italy and France. In 1654, a French nobleman asked the mathematician Blaise Pascal (1623–1662) a question about a gambling game. Pascal wrote to another mathematician, Pierre de Fermat (1601?–1665), initiating a correspondence in which they solved gambling problems and developed probability theory (Figure 13–1 on page 707). In one letter, Pascal wrote [the French nobleman] "is very intelligent, but he is not a mathematician; this as you know is a great defect."

You already know Fermat as one of the inventors of coordinate geometry, but who was Blaise Pascal? Pascal's mathematical ability was evident at a young age. He excelled in geometry and wrote a major paper on the subject at the age of 25. But at 27, Pascal, already in poor health, decided to give up mathematics and devote his life to religion. During the rest of his life, he occasionally returned to mathematics. However, because he lived to be only 39, Pascal is known as the greatest "might-have-been" in the history of mathematics, and his best-known works concern religion.

Pascal and Fermat studied probability to understand gambling. People today use probability to estimate the chances of all kinds of repeated events

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Blaise Pascal

David Eugene Smith Collection, Rare Book and Manuscript Library, Columbia University

Pierre de Fermat

Figure 13–1

that have patterns in the long run. Actuaries in insurance companies use probabilities of catastrophes to determine how much to charge for insurance. Testing services design and score standardized tests according to the probabilities of guessing correctly. Meteorologists use past frequencies of rain to predict the chance of rain tomorrow. Gambling casinos and some state governments use probabilities to design gambling games and lotteries.

13.1 Experimental and Theoretical Probability

NCTM Standards

- collect data using observations, surveys, and experiments (3–5)
- describe events as likely or unlikely and discuss the degree of likelihood using such words as certain, equally likely, or impossible (3–5)
- predict the probability of outcomes of simple experiments and test the predictions (3–5)

Students first encounter probability when they learn how various events are "certain," "likely," "unlikely," or "impossible."

LE 1 Opener

Discussion

Suppose a bag contains 8 blue marbles and 2 black ones. You are going to pull 1 marble out of the bag. Describe an event (result for the color of the marble) that is

(a) likely. **(b)** unlikely. **(c)** impossible. **(d)** certain.

Outcomes and Sample Spaces

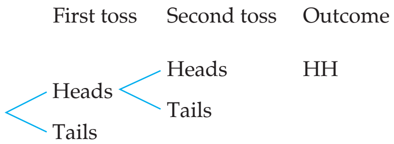
Next, students learn to list all the possible results of an experiment. For example, when you flip a coin, there are two possible results: heads (H) and tails (T). Each possible result, H or T in this case, is an outcome. The set of all possible outcomes, $\{H, T\}$, is called the **sample space** of an experiment.

LE 2 Skill

(a) Complete the sample space for flipping 2 coins. HH means a head on the first coin and a head on the second coin.

$$\{HH, \_\_\_\_, \_\_\_\_, \_\_\_\_\}$$

(b) Complete the tree diagram that shows the sample space.



Any subset of the sample space, such as $\{HT, TH\}$, is called an **event.** Probability questions deal with the probability of an event.

In a coin flip, a head or tail has the same chance of occurring. The outcomes "heads" and "tails" are **equally likely outcomes.** Probabilities for experiments that have equally likely outcomes are easier to compute. For this reason, it is important to determine whether or not outcomes are equally likely.

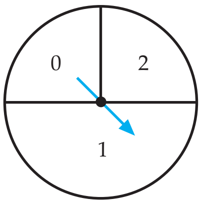


Figure 13–2

LE 3 Concept

(a) Are the outcomes 0, 1, and 2 equally likely to occur on the spinner in Figure 13–2?
(b) Sketch a spinner with regions numbered 0, 1, and 2 so that each outcome is equally likely.

Theoretical Probability

The terms "certain," "likely," "unlikely," and "impossible" describe probabilities in an informal way. Numerical probabilities are more precise measures of the likelihood of events. How do we find numerical probabilities of future events?

If someone asked you for the probability of heads on a coin flip, you wouldn't take a coin out and start flipping it, would you? You would state a theoretical probability. The **theoretical probability** assumes ideal conditions. In the following lesson exercise, find theoretical probabilities for each of the spinners from LE 3.

LE 4 Concept

What is the theoretical probability of spinning a 2 on each of the spinners in LE 3?

LE 4 illustrates **geometric probability** in which the probabilities are proportional to a geometric measurement of such an area. The second spinner has 3 equal areas, so each outcome is equally likely. The probability of each outcome is $\frac{1}{3}$. Theoretical probabilities for an experiment involving equally likely outcomes can be computed as follows.

> **Definition: Probability with Equally Likely Outcomes**
>
> If all outcomes in a sample space S of an experiment are equally likely, the **probability** of an event A is
>
> $$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S}.$$

To use this definition, one must have a sample space that lists equally likely outcomes.

Experimental Probability

We can also approximate some probabilities by conducting experiments or surveys.

LE 5 Concept

(a) You are going to flip a coin 10 times. Predict how many heads you will get.
(b) Flip a coin 10 times and count the total number of heads.
(c) Collect results from the rest of the class and list them.

Your results from LE 5(b) yield an experimental probability for getting heads on a coin toss. For example, if I tossed a coin 20 times and obtained heads seven times, my experimental probability for heads would be 7 out of 20, or $\frac{7}{20}$. The **experimental probability** of A, written $P(A)$, in an experiment of N trials is $\frac{\text{Number of times } A \text{ occurs}}{N}$. An experimental probability gives an estimate of the true (theoretical) probability.

LE 6 Concept

(a) What is your experimental probability for getting heads on a coin toss, based on your results from LE 5(b)?
(b) What is your class's experimental probability for getting heads on a coin toss?

Experimental and Theoretical Probability

How are experimental and theoretical probabilities related? In the following exercise, you will find an experimental probability for a coin flip. Will it come out the same as the theoretical probability?

LE 7 Connection

(a) You want to determine the experimental probability of getting 2 heads when you flip 2 coins. Flip 2 coins 20 times and obtain an experimental probability for 2 heads.

(b) You ask a sixth grade class to determine the theoretical probability of 2 heads. Angela says, "The possible outcomes are {0, heads, 1 head, 2 heads}, so the probability of 2 heads is $\frac{1}{3}$." Bruce says, "The possible outcomes are {*HH*, *HT*, *TH*, *TT*} so the probability of 2 heads is $\frac{1}{4}$." Which student is right? Explain why.

(c) Which of the two spinners in LE 3 is a model for this coin-flipping experiment?

As with coins and spinners, the probabilities of outcomes with regular dice can be determined experimentally or theoretically. Consider, for example, the probability of rolling a sum of 7 on 2 dice (that is, hexahedral random digit generators).

LE 8 Skill

(a) Estimate the probability of rolling a sum of 7 on 2 dice.

(b) Roll the dice 30 times, and make a line plot or bar graph showing the frequency of each sum from 2 to 12.

(c) On the basis of your results, what is the experimental probability for rolling a sum of 7?

(d) If possible, collect results from the whole class. Construct a line plot or bar graph for the class, and determine the class's experimental probability for a sum of 7.

To determine the theoretical probability of rolling a sum of 7 on 2 dice, complete the following lesson exercise.

LE 9 Reasoning

(a) Complete the following list of all possible results for rolling 2 dice.

Sum of 2 Dice

Second Die \ First Die	1	2	3	4	5	6
1	2	3	4	5	6	7
2						
3						
4						
5						
6						

(b) How many (different) possible outcomes are there?

(c) Are they equally likely?

(d) The theoretical probability of rolling a sum of 7 is ________.

(e) How does your answer to part (d) compare to the experimental probability obtained by the class?

LE 10 Skill

(a) Use your table of possible dice outcomes to list the theoretical probability of each sum.

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability											

(b) What is the pattern in your answers to part (a)?
(c) Predict how many sums of 11 you would get if you rolled 2 dice 50 times.

To predict how many times an event will occur, multiply the probability of the event times the total number of trials. In LE 10, $\frac{1}{18} \cdot 50 = 2.\overline{7} \approx 3$ times.

You have been finding both experimental and theoretical probabilities in a variety of situations. What is the relationship between these two types of probabilities?

Discussion

LE 11 Connection

Suppose you select 1 card at random from a regular deck of 52 cards. The theoretical probability of selecting the ace of spades is $\frac{1}{52}$. Which of the following conclusions does this probability lead you to make?

(a) If I repeat this experiment 52 times, I will pick an ace of spades 1 time.
(b) If I repeat this experiment 520 times, I will pick an ace of spades 10 times.
(c) If I repeat this experiment a large number of times, I *will* pick an ace of spades $\frac{1}{52}$ of that number of times.
(d) As I repeat this experiment more and more, the probability of picking an ace of spades sets closer to $\frac{1}{52}$.

The correct statement in the preceding exercise contains two essential parts: "repeat . . . more and more," and "closer." First, the theoretical probability predicts the trend of "more and more" trials. Second, the experimental results get "closer" to the theoretical probability, but may not match it exactly.

A theoretical probability has two interpretations. It tells how likely an event is. It also describes, on average, what happens in the long run. The following statement describes the relationship between the theoretical and experimental probabilities of an event. It is called the "Law of Large Numbers."

Theoretical and Experimental Probability

If the theoretical probability of an event is $\frac{a}{b}$, then $\frac{a}{b}$ approximates the fraction of the time this event is expected to occur when the same experiment is repeated many times under uniform conditions.

It's amazing that the overall results of a *large number* of coin flips are fairly predictable, but an individual coin toss is completely unpredictable! That is why it is both fair and unpredictable to toss a coin at the beginning of a football game to decide who gets the ball.

In some situations, such as weather forecasting, there is no theoretical probability. A probability of rain tomorrow, such as 80%, is an experimental probability. The meteorologist looks at past records of similar weather conditions and sees that it rained the following day about 80% of the time.

LE 12 Reasoning

A principal is expecting a visitor 2 weeks from today who will evaluate the school. She wants to estimate the probability that all her teachers will be in school that day. How can she do it?

A survey is another common way to obtain experimental probabilities.

LE 13 Reasoning

Devise a plan, and solve the following problems. A survey of 1,000 elementary-school teachers regarding their preferred subject areas yielded the following results.

	Math/Science	**Language/History**
Male	14	22
Female	212	752

On the basis of this survey, estimate the probability of each of the following.

(a) An elementary-school teacher is female.
(b) A male elementary-school teacher prefers the math/science area. (*Hint:* Only consider male elementary-school teachers.)
(c) A female elementary-school teacher prefers the math/science area.

A Game: Dice Products

LE 14 Reasoning

Dice Product is a game for two players that uses two dice.

Player 1 rolls both dice and computes the product. Player 2 rolls one die and squares the number. The player with the higher number wins.

(a) Guess whether this is a fair game. If it is not, which player do you think has the advantage?
(b) Play 15 games, and keep a record of who wins each time.
(c) Answer part (a) again. Construct sample spaces to support your answer.

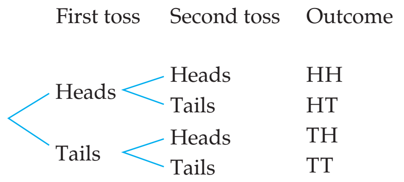
LE 15 Summary

Tell what you learned about theoretical and experimental probability in this section. How are the experimental and theoretical probability of an event related?

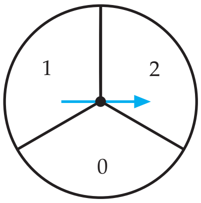
Answers to Selected Lesson Exercises

1. (a) A blue marble
(b) A black marble
(c) A red marble
(b) A marble

2. (a) *HT*; *TH*; *TT*
(b)



3. (a) No, 1 is more likely than 0 or 2.
(b)



4. $\frac{1}{4}$ and $\frac{1}{3}$

6. (a) The number of heads you got divided by 10.

7. (b) Bruce is right. His sample space shows equally likely events, while Angela's does not. The event "1 head" is more likely than 0 or 2 heads because there are two ways to do it.
(c) The spinner from part (a)

9. (a)

Second Die \ First Die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(b) 36 (c) Yes (d) $\frac{6}{36} = \frac{1}{6}$

10. (a)

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(c) About 3

11. (d)

12. She can look at attendance records for the past month of school days and see what percent of the days all her teachers were in school.

13. (a) $\frac{964}{1000} = 0.964$ (b) $\frac{14}{36} = 0.389$
(c) $\frac{212}{964} = 0.22$

13.1 Homework Exercises

Basic Exercises

1. Suppose you roll a regular die. Describe an event that is
(a) likely. (b) unlikely.
(c) impossible. (d) certain.

2. Suppose you pick a card from a regular deck of cards. Describe an event that is
(a) likely. (b) unlikely.
(c) impossible. (d) certain.

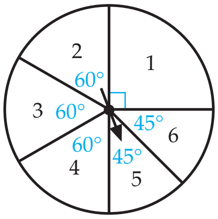
3. You roll a fair die and read the result.
(a) What is the sample space?
(b) Is each outcome equally likely?

4. In an experiment, people rate 3 brands of orange juice, *A*, *B*, and *C*, from best to worst. What is the sample space?

5. Tell whether each experiment has equally likely outcomes:
(a) You shoot a basketball. You make or miss the shot.
(b) You get in your car. It starts or does not start.
(c) You guess on a true-false question. Your answer is right or wrong.

6. Tell whether each experiment has equally likely outcomes:
(a) A baby is born. It is a boy or a girl.
(b) You go to math class. You are on time or late.
(c) You roll a regular die. The result is 1, 2, 3, 4, 5, or 6.

7. A certain experiment consists of spinning a spinner like the one shown.



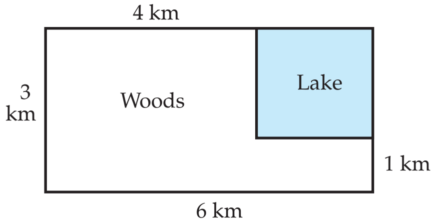
(a) What is the probability of spinning each of the following?
(i) 1 (ii) 2 (iii) 5 (iv) An even number
(b) Suppose you spin the spinner many times. Draw a bar graph showing the relative frequencies of 1, 2, 3, 4, 5, and 6 that you might obtain.

8. Suppose you roll a regular die many times. Draw a bar graph showing the relative frequencies of 1, 2, 3, 4, 5, and 6 that you might obtain.

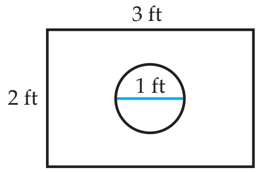
9. In In-Between, you are dealt 2 cards from a standard deck of 52 cards. Then you pick a third card from the deck. In order for you to win, its value must be between the values of the other 2 cards. What is the probability of winning if you are dealt each of the following?
(a) A 2 and a 9 (b) A 3 and a 7

10. A fourth grader has two bags. One bag has 4 red grapes and 2 green grapes. The second bag has 20 red grapes and 10 green grapes. You ask her if she selects one grape at random, which bag gives her the greater chance of picking a red grape. She says the second bag because it has more red grapes. What would you tell the student?

11. A missing airplane crashed somewhere in the region shown. What is the probability that it is in the lake?



12. Suppose an object is dropped at random on the area shown. What is the probability it will land in the circle?



13. Draw a spinner that has four outcomes in which three are equally likely and the fourth has a probability that is three times that of each of the other three.

14. On the basis of the histogram in Figure 12–5 on page 651, what is the probability that a family has an income between \$71,000 and \$105,000?

15. A sixth grader says that when you roll 2 dice, the sum can be 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12. There are 11 possibilities. So $P(\text{sum is } 7) = \frac{1}{11}$. What is wrong with this reasoning?

16. A sixth grader says that when you flip three coins, the possible outcomes are 0 heads, 1 head, 2 heads, and 3 heads. Therefore, the probability of 0 heads is $\frac{1}{4}$. What is wrong with this reasoning?

17. You roll a regular die 5 times. Which result is more likely?
(a) 1 1 1 1 1 (b) 1 2 3 4 5
(c) Both results are equally likely.

18. A fair coin is tossed 5 times. Two possible results are *HHHHH* and *HTHTH*. Which is more likely to occur?
(a) *HHHHH* (b) *HTHTH*
(c) Both are equally likely.

19. The following exercise requires flipping 3 coins.
(a) First, guess what the results will be. Then flip 3 coins 20 times and record the results.

Result	**Number of Times**
3 heads 0 tails	
2 heads 1 tail	
1 head 2 tails	
0 heads 3 tails	

(b) On the basis of part (a), what is your *experimental* probability for getting exactly 2 heads?
(c) List the different possible equally likely outcomes when flipping 3 coins all at once.

First Coin	**Second Coin**	**Third Coin**

(d) What is the *theoretical* probability of getting exactly 2 heads?
(e) How do your answers to parts (b) and (d) compare?

20. (a) Guess how many cards you would have to select from a regular deck of cards (without replacement) to obtain 2 cards of the same denomination (for example, two 7s).
(b) Try it 20 times with a deck of cards and obtain an experimental probability.

21. (a) Guess the probability that a thumbtack lands point up.
(b) Devise an experiment and obtain an experimental probability.

22. Have you ever spun a penny? Stand a penny on its side holding the top with a finger of one hand. Now flick the side of the coin with a finger of your other hand so it spins.
(a) If you spin it 20 times, guess how many heads and tails you will get.
(b) Spin the penny 20 times, and give your experimental probability for heads.

23. Suppose you roll two regular dice and subtract the larger number minus the smaller or the difference of the two numbers if they are equal.
(a) Construct a sample space of equally likely outcomes.
(b) Roll the two dice 20 times, and graph your results with a line plot or bar graph.

24. Open this book to any page and find the first complete paragraph. Use that paragraph to estimate the probability that a word begins with a consonant.

25. Two dice are numbered 1, 2, 3, 4, 5, and 6 and 1, 1, 2, 2, 3, and 3, respectively.
(a) Find the probability of rolling each sum from 2 through 9.
(b) If you rolled these dice 100 times, predict how many times you would roll a sum of 3.

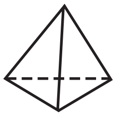
26. (a) What is the probability of rolling a *product* less than 9 on two regular dice?
(b) If you rolled the dice 80 times, predict how many times you would roll a product less than 9?

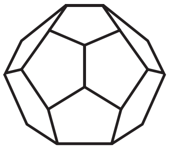
27. Write a sample space for flipping a coin 3 times, and find the probability of getting *at least* 2 heads ("at least 2 heads" means 2 or more heads).

28. Assume that the probability of a pregnant woman's having a baby boy is 50%. If she has 3 children, what are the chances of having 2 boys and 1 girl in any order? (*Hint:* See the preceding exercise.)

29. Everyone has a combination of two alleles for eye color—BB, Bb, or bb, where B is the dominant brown-eye allele and b is the recessive blue-eye allele. A person with at least one brown-eye allele will have brown eyes.
(a) John has Bb alleles, and his wife Mary has Bb alleles. If their child receives one allele at random from each parent, write a sample space of equally likely outcomes for the two alleles of the child.
(b) What is the probability that John and Mary's child will have brown eyes as they do?
(c) Craig has BB alleles, and his wife Lynda had Bb alleles. What is the probability that their child will have brown eyes?

30. In Chapter 8, you were introduced to regular polyhedrons. Usually, dice are shaped like cubes, but any of the regular polyhedrons could be used.


Tetrahedron


Dodecahedron

(a) Each of two tetrahedral dice has an equal chance of rolling a 1, 2, 3, or 4. What is the probability of rolling a sum of 6 on the 2 tetrahedral dice?
(b) The faces of 2 dodecahedrons are numbered 1 through 12. What is the probability of rolling a sum of 6 on 2 dodecahedral dice?

31. You have vulnerable backgammon men 4 and 6 spaces away from your opponent's pieces. What are your opponent's chances of knocking at least one of your pieces off on the next roll of the dice? (Your opponent needs a sum of 4 or 6 on the two dice or a 4 or a 6 on either of the two dice.)

32. Mason wants to bet Juwan that he can roll a product of 8 on two regular dice more often than Juwan can roll a product of 12. Juwan asks your advice about whether to play. Write a note to Juwan explaining whether or not he should accept the bet and explain why.

33. (a) In flipping a coin, why does P (heads) $= \frac{1}{2}$?
(b) If I flip a coin 100 times, how many heads will I get? Tell what is wrong with each of the following students' answers.

Isaac says "I will get 50 heads."
Maya says "I will get close to 50 heads."

(c) Give a better answer to the question in part (b).

34. A sixth grader flips a coin 10 times and obtains 6 heads and 4 tails. She says there is something wrong with her coin because she should have gotten 5 heads and 5 tails. How would you respond?

35. A meteorologist reports that the chance of rain tomorrow is 60%. What does this mean?

36. How would you find the experimental probability of your mathematics professor continuing to teach beyond the end of the period?

37. The following table shows the results of a survey of 1,000 buyers of new or used cars of a certain model.

	Satisfied	Not Satisfied
New	300	200
Used	220	280

On the basis of this survey, what is the probability of each of the following?
(a) A new car buyer is satisfied.
(b) A used car buyer is satisfied.
(c) Someone who is not satisfied bought a used car.

38. A survey of 400 community college students yielded the following results.

	Democrat	Republican	Other
Freshmen	95	70	40
Sophomores	86	84	25

On the basis of this survey, what is the probability of each of the following?
(a) A college student is a Republican.
(b) A freshman is a Democrat.
(c) A sophomore is a Democrat.

Extension Exercises

39. Pascal's triangle relates to a variety of mathematics problems.

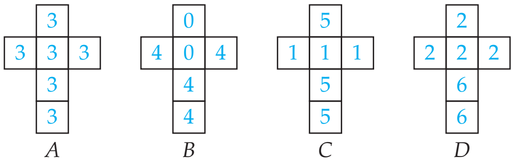
1 1
1 2 1
1 3 3 1
1 4 6 4 1

(a) Add a fifth row, continuing the same pattern.
(b) In flipping 1 coin, there is 1 way to get 0 heads and 1 way to get 1 head. In flipping 2 coins, how many ways are there to get 0 heads? 1 head? 2 heads?
(c) How does part (b) relate to Pascal's triangle?
(d) Explain how the third row of the triangle relates to flipping 3 coins.
(e) Using Pascal's triangle and the answer to part (c) to solve part (d) involves ________ reasoning.

40. (a) $(x + y)^1 =$ ________
(b) $(x + y)^2 =$ ________
(c) How are the results of parts (a) and (b) related to Pascal's triangle?
(d) Compute $(x + y)^4$ using Pascal's triangle.

41. Sicherman described the only nonstandard pair of cubical dice with counting numbers on each face that have the same probabilities for the sums 2 to 12 as regular dice. Can you figure out what numbers are on the faces? (*Hint:* The two dice have different numbers. One of them has only counting numbers from 1 to 4, inclusive.)

42. WIN, designed by Bradley Efron, is a game that uses unusual cubical dice. The faces are shown here.



You pick 1 die, and your opponent picks 1 die. Whoever rolls the higher number wins.

(a) Suppose you pick *A*. Which die would a smart opponent then pick?

(b) Suppose your opponent picks *B* first. Which die would you then pick?

(c) Suppose your opponent picks *C* first. Which die would you then pick?

43. What is the probability of rolling a sum of 6 on three regular dice?

44. At the Rockville train station, trains run hourly in each direction. If you arrive at the train station at a random time, the next train is 3 times more likely to be northbound than southbound. Explain how this can be the case.

Technology Exercise

45. Go to www.shodor.org/interactive/activities and try out the activity called "Coin Toss." Would you use this activity with students in elementary school?

Project

46. Read "Inflexible Logic" by Russell Maloney in *Fantasia Mathematica*, edited by Clifton Fadiman. Write a report that includes a summary of the story and your reaction to it.

Video Clip

47. Go to www.learner.org and watch "Teaching Math: A Video Library 5-8" video #31. Fourth graders roll dice and study experimental and theoretical probability. Discuss the lesson. Name three specific strengths of the lesson and one way the lesson might be improved.

13.2 Probability Rules and Simulations

NCTM Standards

- understand the measure of likelihood of an event that can be represented by a number from 0 to 1 (3–5)
- understand and use appropriate terminology to describe complementary and mutually exclusive events (6–8)
- use proportionality and a basic understanding of probability to make and test conjectures about the results of experiments and simulations (6–8)

Probability has its own set of mathematical rules. For example, most numbers cannot represent probabilities. How large or small can a probability be? Next, consider arithmetic of probabilities. Under what circumstances can we add or subtract probabilities of events? (Multiplication of probabilities is covered later in the chapter.)

This section also presents another method for obtaining experimental probabilities, called simulation. Simulations are used because they are easier to perform than the actual experiments.

Probability Values

What numbers can represent probabilities? Read the cartoon in Figure 13–3, and then try LE 1.

Figure 13–3

LE 1 Opener

What numbers can we have for probabilities?

(a) When you listen to the weather report, what could the probability of rain next Tuesday be? Give all possible answers.

(b) Part (a) suggests that all probabilities are between ________ and ________ (inclusive).

LE 1 illustrates why probabilities can be as low as 0 or as high as 1. A probability tells what percent of the time an event is likely to happen, so it could be any number between 0% (or 0) and 100% (or 1). In the equally-likely-outcomes formula

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S}$$

the number of outcomes in A can be as low as 0, making $P(A) = 0$, or as high as the number of outcomes in S, making $P(A) = 1$.

Probability Values of an Event

If A is any event, then $0 \leq P(A) \leq 1$.

Probabilities are normally expressed as fractions, decimals, or percents. (Odds and ratios are covered in Section 13.5.) For example, if I flip a coin, the probability of heads is given by

$$P(\text{heads}) = \frac{1}{2} = 0.50 = 50\%$$

LE 2 Concept

Which of the following could not be the probability of an event?

(a) $\frac{2}{3}$ **(b)** -3 **(c)** $\frac{7}{5}$ **(d)** 15% **(e)** 0.7

In Section 13.1, you classified probabilities of events as "impossible," "unlikely," "likely," or "certain." How are these words related to numerical probabilities? Figure 13–4 shows possible probabilities of events and some corresponding verbal descriptions.

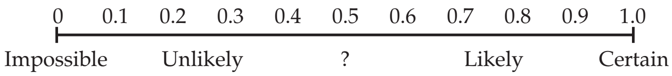


Figure 13–4

Mutually Exclusive Events

How are probabilities that involve more than one event computed? One of the simplest cases involves two events that do not intersect.

LE 3 Concept

Pascal College has 25% freshman, 25% sophomores, and 25% women.

(a) What is the probability that a student is a freshman *and* a sophomore?
(b) Name two nonintersecting groups at the college besides freshmen and sophomores.
(c) What is the probability of selecting a single student who has both of the characteristics you mentioned in part (b)?

LE 3(a) and (c) suggest the definition of nonoverlapping—that is, mutually exclusive—events. Mutually exclusive events cannot occur at the same time. Events A and B are **mutually exclusive** or **disjoint** if and only if $P(A \text{ and } B) = 0$ or $A \cap B = \phi$. The expression "$P(A \text{ and } B)$" means the probability that both events A and B occur.

$P(A \text{ or } B)$ is easier to compute for mutually exclusive events than for other events.

Classroom Connection

LE 4 Concept

Pascal College has 25% freshman, 25% sophomores, and 25% women. A student is picked at random.

(a) Which of the following are mutually exclusive groups?
(1) freshmen and sophomores (2) freshmen and women
(b) What is the probability that a randomly chosen student is a freshman or a sophomore? Draw a rectangular diagram to support your answer.
(c) A sixth grader says that the probability of choosing a freshman woman is 50%. Is this right? If not, what would you tell the student?

In part (b), you could add the probabilities (percentages) because the two groups, freshmen and sophomores, do not overlap. When two events A and B do overlap, as in part (c), you cannot simply add their probabilities to compute $P(A \text{ or } B)$.

The general rule for $P(A \text{ or } B)$ when A and B are mutually exclusive is as follows.

The Addition Rule for Mutually Exclusive Events

If A and B are mutually exclusive events, $P(A \text{ or } B) = P(A) + P(B)$.

The expression "A or B" combines all the outcomes from event A and event B. For this reason, $P(A \text{ or } B)$ can also be written as $P(A \cup B)$. However, current middle-school textbooks do not use set symbols in probability.

Probabilities have already been defined for equally likely outcomes. For a sample space with outcomes that are not equally likely, the addition rule can also be used to compute probabilities.

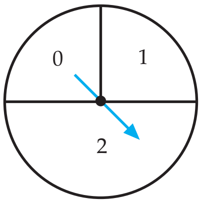


Figure 13–5

LE 5 Skill

Show how to determine the probability of spinning a 1 or a 2 on the spinner in Figure 13–5.

The preceding exercise illustrates the following.

Definition: The Probability of an Event

If A is any event, $P(A)$ is the sum of the probabilities of all outcomes in set A.

A set of outcomes is mutually exclusive. This is why we can add the probabilities of the outcomes. The addition rule for mutually exclusive events can also be used to derive a formula for events that are complements.

Complementary Events

The probability that it rains tomorrow has a simple relationship to the probability that it doesn't rain tomorrow. The event "not rain" is the complement of the event "rain." The **complement** of any event A is the event that A does *not* occur, written "not A" or $\overline{A}$. (Some textbooks use the notation A'.)

LE 6 Reasoning

(a) Suppose that the probability of rain tomorrow is $\frac{1}{4}$. What is the probability that it will *not* rain tomorrow?

(b) If the probability of an event is n, what is the probability of the complement of the event?

LE 6(b) is about the generalization that the probability of the complement of an event is 1 (or 100%) minus the probability of the event.

Probabilities of Complementary Events

$$P(\overline{A}) = 1 - P(A)$$

The property of complementary event probabilities follows from the addition rule for mutually exclusive events.

LE 7 Reasoning

(a) A and $\overline{A}$ are mutually exclusive. By the addition rule, $P(A \text{ or } \overline{A}) =$ _________.
(b) What is the numerical value of $P(A \text{ or } \overline{A})$?
(c) Combine the equations in parts (a) and (b).
(d) How would you derive the probability rule for complementary events from your equation in part (c)?
(e) Do parts (a)–(d) involve induction or deduction?

Complementary events have no outcomes in common (are mutually exclusive), and together they encompass all possible outcomes. In set notation, $A \cap \overline{A} = \phi$, and $A \cup \overline{A} = S$, the sample space.

Simulations

When the theoretical probability is difficult or impossible to compute, and it is impractical to find an experimental probability, a simulation can be designed. A **simulation** is a probability experiment that has the same kind of probabilities as the real-life event.

For example, a series of coin flips can be used to simulate whether a series of newborns will be boys or girls. The result of a coin flip has the same kind of probabilities as the sex of a baby. The probability of heads is $\frac{1}{2}$ and the probability of tails is $\frac{1}{2}$, just as the probability of a boy is about $\frac{1}{2}$ and the probability of a girl is about $\frac{1}{2}$. Furthermore, each coin flip has no effect on subsequent coin-flip probabilities. The same is usually true of the sexes of babies.

In the following simulation, each coin flip represents the birth of a new child. Furthermore, suppose that each head represents having a girl and each tail represents a boy.

LE 8 Connection

My friends would like to have a baby girl. They are willing to have up to 3 children in an effort to have a girl.

(a) Guess the probability that they will have a girl.
How can you calculate the probability for them? Use a coin-flipping simulation to approximate the answer. Form pairs to do this.
(b) Using $H =$ girl and $T =$ boy, simulate the event of the couple's having a family. (For example, if you flip heads on the first try, they have 1 girl and stop having children.) What family do you end up with?
(c) Repeat this experiment 9 more times so that you have created 10 families.
(d) Out of the 10 families, how many have a girl?
(e) On the basis of your results, estimate the probability of the family's having a girl.
(f) If you have data from others in the class, use these data to revise your answers to part (e).

A simulation is usually much easier to perform than the event it represents. The results of a simulation provide an experimental probability for the simulated event. A greater number of trials is likely to provide a better estimate.

A company is buying 8 automobiles from a manufacturer. From past experience, they know that about 1 out of every 6 automobiles needs some kind of adjustment. You can simulate determining whether or not automobiles need adjustment by rolling a die or by using a calculator or computer program.

Discussion

LE 9 Connection

Suppose you buy 8 automobiles, and you expect that about 1 out of every 6 will need an adjustment. What is the probability that 2 or more automobiles will need adjustments?

Use a die to simulate 10 purchases of 8 automobiles like the one just described.

(a) What roll or rolls on the die will represent an automobile needing adjustment?

To perform the simulation, choose part (b) if you want to roll a die, part (c) if you want to use a random number table, parts (d)–(f) if you want to use a computer spreadsheet, or part (g) if you want to use a graphing calculator.

(b) Roll the die to simulate 10 orders of 8 cars each, and record your results. Proceed to part (h).

(c) Use the following random number table or generate your own random numbers.

74192	77567	88741	48409	41903	43909	99477	25330	64359
40085	16925	85117	36071	15689	14227	06565	14374	13352
49367	81982	87209	36579	58984	68288	22913	18638	54303
00795	08727	69051	64817	87174	09517	84534	06489	87201

Start at the left, and count each digit between 1 and 6 (inclusive) as representing a car. In part (a), you have decided which digit means that the car needs an adjustment. Continue through the random number table until you have finished buying 10 sets of 8 cars. Record your results, and proceed to part (h).

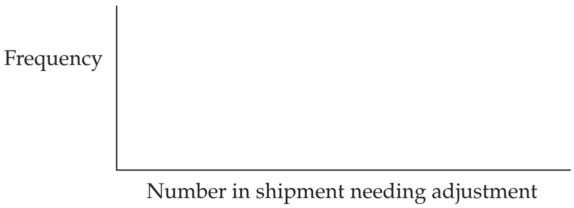
(d) You can generate random numbers with a spreadsheet. In EXCEL, use the command RAND( ) to select a number between 0 and 1. You can generate numbers from other sets with commands such as 6*RAND( ) or INT(6*RAND( )). The INT command rounds decimals down to the nearest integer. The command 6*RAND( ) selects numbers from what set?

(e) The command INT(6*RAND( )) selects numbers from what set?

(f) Write a command and select 10 sets of 8 random whole numbers from 1 to 6. Record your results, and proceed to part (h).

(g) On the TI-83, press MATH and then highlight PRB. Choose randInt(and press ENTER. Then type 1, 6) and press ENTER to choose a random integer between 1 and 6 inclusive. Choose 10 sets of 8 integers. Record your results.

(h) Make a frequency graph of the number of automobiles needing adjustment in each of your 10 shipments.



(i) What is your experimental probability that 2 or more cars in a shipment will need adjustments?

Statisticians often use simulations to approximate probabilities that are difficult to compute using theory or for which there is no theory. Computers offer the possibility of simulating a large number of trials in a short amount of time.

LE 10 Summary

Tell what you learned about probability rules and simulations in this section. Give an example that illustrates each probability rule.

Answers to Selected Lesson Exercises

1. (a) 0% to 100% (inclusive) (b) 0;1

2. (b) and (c)

3. (a) 0 (b) Men and women (c) 0

4. (a) (1) (b) 50%
(c) No. It cannot be determined, because some of the freshmen are also women.

5. $P(1 \text{ or } 2) = P(1) + P(2) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

6. (a) $\frac{3}{4}$
(b) $1 - n$

7. (a) $P(A) + P(\overline{A})$
(b) 1
(c) $P(A) + P(\overline{A}) = 1$
(d) Subtract $P(A)$ from both sides of the equation.
(e) Deduction

9. (e) {0, 1, 2, 3, 4, 5}

13.2 Homework Exercises

Basic Exercises

1. Which of the following numbers cannot be a probability?
(a) 0.6 (b) −1 (c) 3 (d) 0% (e) $\frac{1}{1000}$

2. An event is very unlikely to happen. Its probability is about
(a) $\frac{1}{10}$ (b) $\frac{3}{10}$ (c) $\frac{1}{2}$ (d) $\frac{7}{10}$ (e) $\frac{9}{10}$

3. Determine whether or not each pair of events is mutually exclusive.*
(a) Getting an A in geometry and a B in English this semester
(b) Getting an A in geometry and a B in geometry this semester

4. Suppose that you flip a coin once. Name two events, A and B, that are mutually exclusive.

5. The probability that a person has black hair is 22%. The probability that a person has brown eyes is 72%. A student says that the probability that a person has either black hair or brown eyes is 94%. Is this right? If not, what would you tell the student?

6. A student says, "Because there are 7 continents, the probability of being born in North America is $\frac{1}{7}$." Is this right? If not, what would you tell the student?

7. At a college, the probability that a randomly selected student is a freshman is 0.35, and the probability that the student is a sophomore is 0.2. What is the probability that a randomly selected student is neither a freshman nor a sophomore?

*For more practice, go to www.cengage.com/math/sonnabend

8. Suppose Gordon Gregg and Debra Poese are the only two candidates for president. The probability that Gordon wins is 0.38. What is the probability that Debra wins?

9. Jane Austen College has 1,020 male and 1,180 female students. What is the probability that a student selected at random will be female?

10. A school has a raffle for the 130 students in the fifth and sixth grades. There are 62 fifth graders. What is the probability that the grand-prize winner selected at random from 130 students will be a sixth grader?

11. Use a coin flip or a computer program to simulate the birth of a boy or girl. Investigate four-child families by doing the following.

(a) If there are 20 four-child families, guess how many will have exactly 1 girl.

(b) Create 20 four-child families, using $H =$ girl and $T =$ boy. Write down the result for each family.

(c) Use your results to complete the table.

Four-Child Families

Number of girls	0	1	2	3	4
Estimated probabilities					

(d) Construct a histogram from your table.

(e) Of the 20 families, how many have exactly 1 girl?

12. Suppose the chance of rain for each of the next three days is 50%. You are planning an outdoor activity and want to estimate the chance that it will not rain at least one of the days.

(a) Guess the probability that it will not rain at least one of the three days.

(b) Use a coin or a random number generator to simulate the weather each day. Make 20 sets of three days and record the results.

(c) Make a histogram of the frequency of 0, 1, 2, 3 days with rain.

(d) What is the experimental probability that it will not rain at least one of the three days?

13. New Raisin Crumbles come with 1 of 6 different famous-mathematics-teacher cards in each box.

(a) Guess the approximate number of boxes you would have to buy to get all 6 different teacher cards.

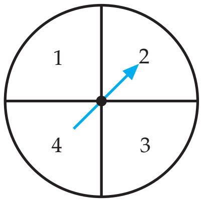
(b) Using a die, simulate collecting all of the cards 8 times to obtain an experimental estimate.

(c) Find the mean and the standard deviation of your data.

(d) Make a box-and-whisker plot of your results.

14. Consider the following problem: "On a game show, a contestant gets to pick box 1, 2, or 3. One box contains $10,000. The other 2 boxes are empty. Use a die to simulate 10 groups of 5 shows (weekly), and estimate the probability that at least 4 people will win during a particular 5-show week." Devise a plan and solve the problem.

15. Two contestants on a game show must choose door 1, 2, 3, or 4. Three of the locations have prizes, and the other does not. A simulation is carried out with a spinner to find the probability that both contestants will win a prize. It is assumed that doors 1, 2, and 4 lead to prizes and door 3 is a dud. The results follow.



1	3
2	3
4	2
1	4
2	4

(a) What does the first row of numbers represent?

(b) On the basis of these results, what is the probability that both contestants will win a prize?

16. A particular forward on the Phoenix Suns makes 50% of his shots. Using a coin, someone simulates the player taking 6 shots. *H* (heads) means that he made the shot and *T* (tails) means that he missed the shot. On the basis of the following results, what is the experimental probability that the player makes exactly 3 out of 6 shots?

H H T T H T
H T T T T H
H T H T T T
H H T T H H
H H H H H T
H T T T H H
H T T H T T

17. Twenty people are at a party.
(a) Guess the probability that at least 2 people will have the same birthday.
(b) Use a random number table or technology to generate 20 random integers between 1 and 365 (inclusive), and see if at least 2 match. Repeat this 25 times to obtain an experimental probability.

18. You own two National Motors cars. Suppose that on a given morning, the probability that the Squealer starts is 0.4 and the probability that the Alley Cat starts is 0.2. Use a random number table or generator to estimate the probability that at least one of your cars will start.
(a) Simulate trying to start both cars 20 times.
(b) Estimate the probability that at least one car will start.

Extension Exercises

19. Draw a spinner, with sectors labeled 2, 3, 4, . . . , 12, that could be used to simulate rolling 2 dice and computing the sum.

20. Draw a spinner that has all the following characteristics.

(1) The probability of spinning a 2 is $\frac{1}{2}$.

(2) The probability of spinning an odd number is $\frac{1}{4}$.

(3) The probability of spinning a sum of 5 on 2 spins is $\frac{1}{8}$.

21. You are hired to determine how long a traffic signal should stay green in each direction at a particular intersection. First, consider the cars going from east to west (on a one-way street).
(a) Suppose you find out that 200 cars pass by during a 10-minute period. One car passes east to west every __________ seconds.
(b) Using a timer, you also compute that cars leave the intersection at a rate of 1 per second when the light is green. What if you make the light alternately red and green every 15 seconds (disregard yellow)? The east-west traffic pattern can be simulated with a die. This will give you an idea of how many cars will get backed up at the traffic light.

(Continued in the next column)

One car passes every 3 seconds. If each roll of the die represents 1 second, decide which die results represent "car" and which represent "no car," and write this information in the table.

	Result on Die
Car	
No car	

(c) Now for the simulation. Start with a red light, and assume that 1 car can leave the intersection each second. Roll 1 die 120 times (for 120 seconds), and fill in the results in a table like the one shown here. Compute how many cars are lined up each second.

Second	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**. . .**
Result on die											
New car arrives?											
Car leaves? (when light is green)											
Total number of cars lined up											

(d) What was the largest number of cars you had backed up? Is this result satisfactory? If not, suggest an alternative timing for the signal.

22. The following supermarket-checkout simulation is adapted from NCTM's "Student Math Notes" of March 1986.

You decide to open a supermarket. How many checkout lanes should you have? A die will be used to simulate the arrival of shoppers at the checkout counter. Assume that a new customer arrives at the checkout counter during 1 out of every 3 minutes. Assume that it takes the cashier 3 minutes to process each customer.
(a) What die results would represent a customer arriving during a given minute?
(b) What die results would represent a customer not arriving during a given minute?

(Continued on the next page)

(c) Roll the die 30 times and simulate 30 minutes at a checkout counter (or use a computer program). See how long the line gets. You could use a chart like the following one to keep a record.

END OF MINUTE	1	2	3	4	5	6	. . .
Customer arrives	*A*	—	—	*B*	*C*		
Checking out	—	*A*	*A*	*A*	*B*	*B*	
Waiting	—	—	—	—	—	*C*	

(d) Complete the following record of your results.

Customer	*A*	*B*	*C*	*D*	*E*	*F*	. . .
Minute of checkout arrival							
Minute checkout completed							
Minutes wait before checkout began							

(e) Give the following estimates based on your 30 minutes of data.

Number of customers arriving ______
Total customer waiting time ______
Average waiting time ______
Total clerk idle time ______
Total time to process everyone ______

(f) Repeat parts (c)–(e) using two checkout counters.

(g) Which number of checkout counters works out better? Consider the waiting time and the cost to the store.

23. Create a dice game that simulates baseball. Use probabilities something like the following.

STRIKE OUT 17% WALK 9%
SINGLE 16.5%
DOUBLE 4% TRIPLE 0.5%
HOME RUN 3%
FLY OUT 22%
OUT OR DOUBLE PLAY
(IF RUNNER IS ON FIRST) 6%
GROUNDOUT (RUNNERS ADVANCE
1 BASE) 22%

Project

24. Design a simulation game that uses dice or a spinner.

13.3 Counting

NCTM Standards

- use geometric models to solve problems in other areas of mathematics, such as number and measurement (3–5)
- recognize and use connections among mathematical ideas (pre-K–12)
- recognize and apply mathematics in contexts outside of mathematics (pre-K–12)

To compute theoretical probabilities, one often counts the number of equally likely results in a sample space. Sometimes the sample space is so large that shortcuts are needed to count all the possibilities.

Discussion

LE 1 Opener

How do state officials know how many different license plates can be made using a certain number of letters and digits (Figure 13–6)?

Photo courtesy of Thomas Sonnabend

Figure 13–6

To answer LE 1, people use methods to count all the possible arrangements of a sequence.

Organized Lists and Tree Diagrams

To understand the shortcut for counting possible license plates, we'll begin by analyzing some simpler counting problems.

MM ■ **Example 1** Carla is taking 3 shirts (pink, green, yellow) and 2 pairs of pants (blue, red) on a trip. How many different choices of outfits does she have (assuming everything matches)? (This can be done with "Bobbie Bear" at: illuminations.nctm.org.)

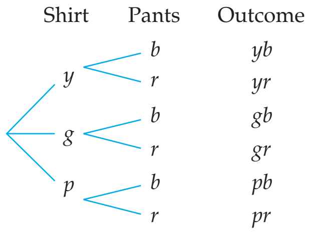
Solution

There are 3 ways to solve this problem. If the shirts are *p*, *g*, and *y* and the pants are *b* and *r*, you can count the outfits in an organized list or a tree diagram (Figure 13–7). A tree diagram can be used when an event has two or more steps. Each step is shown with a set of branches. The third method uses the counting principle model from Section 3.3. You can multiply the number of ways to choose a shirt by the number of ways to choose a pair of pants.

yb *gb* *pb*

yr *gr* *pr*

Organized list



Tree diagram

(3)	×	(2)	=	6
Shirt choice		Pants choice		Outfit choices

Counting principle

Figure 13–7

There are 6 different outfits. ■

MM Discussion

LE 2 Connection

A cafe offers the following menu. You will choose one of two appetizers and one main dish.

LUNCHEON MENU $4

Appetizers

Canned fruit du jour

Cream of bamboo soup

Entrees

Blowfish thermidor

Woodchuck pilaf

Semi-boneless falcon

Hippo in a blanket

(a) Use a tree diagram to show all possible selections you can make. (Abbreviate the item names.)

(b) Use an organized list to show all possible selections you can make.

(c) What is a shortcut for counting the total number of possible orders without using a tree diagram or an organized list?

The Fundamental Counting Principle

Did you recognize the counting principle model of multiplication from Section 3.3? In Example 1 and LE 2, you could multiply the number of ways to make the first choice by the number of ways to make the second choice to obtain the total number of arrangements.

Classroom Connection

LE 3 Concept

Consider the following problem:

"Suppose I want to order a dessert at a restaurant. The menu offers 3 kinds of pie and 2 kinds of cake. How many dessert choices do I have?"

A fifth grader says there are $2 \times 3 = 6$ choices. Is that right? If not, what would you tell the student?

In Example 1 and LE 2, you can use the counting principle, also known as the *Fundamental Counting Principle.* It does not apply to the dessert choice in LE 3.

The Fundamental Counting Principle

If an event M can occur in m ways and, after it has occurred, an event N can occur in n ways, then event M followed by event N can occur in $m \cdot n$ ways.

The Fundamental Counting Principle works not only for two events in sequence but also for any number of events in sequence. The Fundamental Counting Principle can be applied to a complex event if the event can be thought of as a *series of steps* with a specified order. ● Figure 13–8 shows how a seventh-grade textbook uses a tree diagram and the Fundamental Counting Principle on a two-step plan.

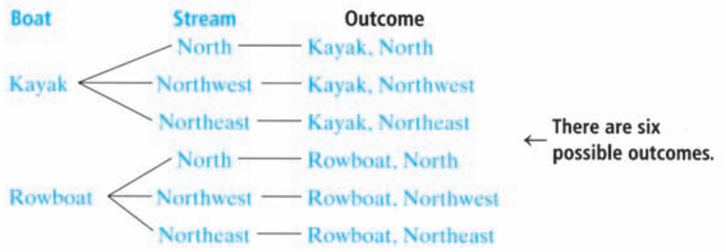
Classroom Connection

EXAMPLE Using a Tree Diagram

2 **River Travel** Suppose you are going to travel on a river. You have two choices of boats—a kayak or a rowboat. You can go upstream on three smaller streams, to the north, northwest, and northeast.

a. What is the sample space for your journey?

Make a tree diagram for the possible outcomes.



b. Suppose you select a trip at random. What is the probability of selecting a kayak and going directly north?

There is one favorable outcome (kayak, north) out of six possible outcomes. The probability is $\frac{1}{6}$.

Quick Check

2. a. Suppose a canoe is added as another choice of boats in Example 2. Draw a tree diagram to show the sample space. See back of book.

b. Find the probability of selecting a canoe at random for the trip. $\frac{1}{3}$

In Example 2 above, there are 2 choices of boats and 3 choices of direction. There are 2 × 3, or 6, total possible choices. This suggests a simple way to find the number of outcomes—using the **counting principle.**

KEY CONCEPTS The Counting Principle

Suppose there are m ways of making one choice and n ways of making a second choice. Then there are $m \times n$ ways to make the first choice followed by the second choice.

Example

If you can choose a shirt in 5 sizes and 7 colors, then you can choose among 5 × 7, or 35, shirts.

Figure 13–8 Tree diagrams in grade 7

Example 2 returns to the question posed in LE 1.

Example 2

The state of Maryland has automobile license plates consisting of 3 letters followed by 3 digits. How many possible license plates are there?

Solution

Understanding the Problem How many ways are there to choose 3 letters followed by 3 digits? A license plate can be created in 6 steps: picking each of the 3 letters and then selecting each of the 3 digits. For example, I might pick FUN 123. There is a certain number of choices for each step.

Devising a Plan Using the Fundamental Counting Principle, one can compute the total number of choices for each step and then multiply these numbers together.

Carrying Out the Plan There are 26 choices for each of the 3 letters and 10 choices for each of the 3 digits. The total number of possible license plates is

$$\underset{\text{Letter}}{(26)}\ \underset{\text{Letter}}{(26)}\ \underset{\text{Letter}}{(26)}\ \underset{\text{Digit}}{(10)}\ \underset{\text{Digit}}{(10)}\ \underset{\text{Digit}}{(10)} = 17{,}576{,}000$$

Looking Back The same technique would work for many other license plate designs.

LE 4 Skill

A state is considering using license plates with 2 letters followed by 4 digits. How many possible license plates can be made?

LE 5 Skill

How many possible telephone numbers can you form if you may choose any 10 digits, except that you may not select 0 or 1 as the first or fourth digit? (*Hint:* Write an example of a phone number, and see how many choices you have for each step.)

One special type of counting problem involves a series of steps in which each step uses up one of the choices.

Example 3

The math club at Pythagoras College has 10 members. In how many ways can the club choose a president and vice-president if everyone is eligible?

Solution

Understanding the Problem Think of the elections as 2 steps. The first step is to elect a president. Then the club elects a vice-president from the remaining members.

Devising a Plan Use the Fundamental Counting Principle.

Carrying Out the Plan How many choices are there for president? 10. Once the president is selected, how many choices are there for vice-president? 9. Using the Fundamental Counting Principle, there are $10 \cdot 9 = 90$ ways to select a president followed by a vice-president.

Looking Back This procedure works for any election in which each position is different.

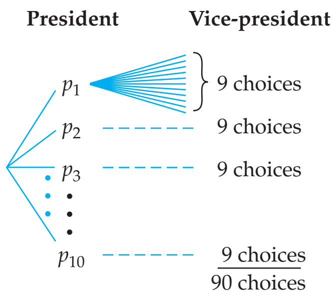


Figure 13–9

An ordered arrangement of people or objects (such as a choice of president and vice-president) is called a **permutation.** In the preceding example, the number of permutations is 90, as shown by the tree diagram in Figure 13–9. There are 10 branches for the first step, and each of these 10 choices has 9 branches for the second step ($10 \cdot 9 = 90$).

LE 6 Skill

The principal of a school plans to select a head teacher, an assistant head teacher, and a workshop coordinator from her faculty of 30 people. In how many ways can she do this?

You can compute the number of permutations in Example 3 and LE 6 with a formula. In picking 2 people from 10, you compute $10 \cdot 9$. In picking 3 teachers from 30, you compute $30 \cdot 29 \cdot 28$. In each product, the first factor is the overall number of people. The number of factors is the number of people you are picking. More generally, the number of permutations or r object from n objects is $n \cdot (n - 1) \cdot \ldots \cdot (n - r + 1)$.

Permutation Formula

If n objects are chosen r at a time, the number of permutations (ordered arrangements) is

$$_nP_r = \overbrace{n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot (n - r + 1)}^{r \text{ steps}}$$

Try out this formula on the permutation problem on LE 6.

LE 7 Skill

(a) Solve LE 6 using the permutation formula. (*Hint:* First find the values of n and r.)

(b) To solve the same problem with a calculator, first enter the value of n. Next, press the permutation key if your calculator has one. On the TI-83, press MATH and highlight PRB. Then choose nPr. Finally, on either type of calculator, enter the value of r.

Finding Probabilities Using the Fundamental Counting Principle

People who design multiple-choice tests must figure out the probability that an examinee will guess a correct answer. A test is not very useful if someone who knows nothing about the material does well on it.

Suppose you give a 5-question multiple-choice quiz with 4 choices for each question. What is the probability that someone who randomly selects an answer from each set of choices will do well (that is, will get at least 4 out of 5 questions right)?

Example 4 shows how to solve part of this problem. You can complete the solution in the exercise that follows Example 4. You will need to use the Fundamental Counting Principle on various sets of test responses and then *add* together the number of ways of doing each set of responses.

■ **Example 4** A quiz has 5 multiple-choice questions. Each question has 4 answer choices, of which 1 is the correct answer and the other 3 are incorrect. Suppose that you guess all the answers.

(a) How many ways are there to answer the 5 questions?
(b) What is the probability of getting all 5 questions right?
(c) What is the probability of getting exactly 4 questions right and 1 wrong?
(d) What is the probability of doing well (getting at least 4 right)?

Solution

(a) In how many ways can you answer the 5 questions? Consider each question as a step in completing a test. There are 4 ways to do each step. Using the Fundamental Counting Principle,

$$(4)(4)(4)(4)(4) = 4^5 = 1024$$

(b) How many ways are there to get all 5 questions right? There is 1 way to answer each question correctly. Using the Fundamental Counting Principle, $(1)(1)(1)(1)(1) = 1$. There is 1 way to answer all 5 questions correctly out of 1,024 possibilities. So

$$P(\text{all 5 right}) = \frac{1}{1024}$$

(c) There is more than one way to get 4 right and 1 wrong, depending on *which question you get wrong*. In the following table, which lists all possible responses that involve at least 4 right answers, *R* stands for a right answer and *W* stands for a wrong answer.

Each type of response (for example, *WRRRR*) can occur in 3 ways based on the Fundamental Counting Principle. Because each of the 5 sets of responses is a complete test result, we add the five 3s together.

Five Responses	Number of Ways to Fill Out the Test
WRRRR	(3)(1)(1)(1)(1) = 3
RWRRR	(1)(3)(1)(1)(1) = 3
RRWRR	(1)(1)(3)(1)(1) = 3
RRRWR	(1)(1)(1)(3)(1) = 3
RRRRW	(1)(1)(1)(1)(3) = 3
	15 ways

So there are 15 ways out of the 1,024 possible ways that result in 4 right answers and 1 wrong answer.

$$P(\text{4 right, 1 wrong}) = \frac{15}{1{,}024} \approx 1.5\%$$

The chances of getting exactly 4 right by guessing are not very good. It makes more sense to study.

(d) "At least 4 right" means you can get either 4 right and 1 wrong or all 5 right. We add the probabilities (of mutually exclusive events).

$$P(\text{at least 4 right}) = P(\text{4 right, 1 wrong}) + P(\text{5 right})$$
$$= \frac{15}{1{,}024} + \frac{1}{1{,}024} = \frac{16}{1{,}024} \approx 0.016$$

■

People who construct multiple-choice tests know the probabilities that someone will get various numbers of questions right by guessing randomly. To decrease the probability of someone's doing well just by guessing, test constructors use a lot more than 5 questions.

LE 8 Reasoning

You are taking a true/false test with 6 questions.

(a) How many ways are there to answer the 6-question test?
(b) What is the probability of getting at least 5 right by guessing the answers at random?

The preceding example and lesson exercise both required multiplication *and* addition. Multiply numbers that represent the number of ways to do *each step*, such as answering each question on an exam. Add numbers that represent *complete results*, such as how many ways to answer 5 right on the whole test and how many ways to answer 6 right on the whole test.

Permutations and Combinations

In how many different ways can 8 horses finish in a race (assuming there are no ties)? Using the Fundamental Counting Principle, the number is

$$(8)(7)(6)(5)(4)(3)(2)(1) = 40{,}320$$

This is another example of an ordered arrangement called a permutation. Products such as (8)(7)(6)(5)(4)(3)(2)(1) can be written in a shorthand notation called factorial. In this case, $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8!$ (read "8 factorial").

Factorial Notation

$n!$ is called ***n* factorial,** and $n! = n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$, in which n is a positive integer. By definition, $0! = 1$.

The horse race example illustrates that the number of permutations of n objects is $n!$.

LE 9 Skill

(a) Compute 5! by hand.
(b) If your calculator has a factorial key, compute 5! on it.

LE 10 Skill

A group has 4 people: Armando, Belinda, Craig, and Danica. In how many ways can the group choose

(a) a president and a vice-president?
(b) a president, a vice-president, and a treasurer?

Example 5 shows two similar types of problems with different answers.

■ **Example 5** A group has 4 people: Armando, Belinda, Craig, and Danica. In how many ways can the group choose

(a) 2 people for a committee? **(b)** 3 people for a committee?

Solution

(a) This is the same situation as LE 10(a), but there won't be as many possibilities because picking persons A and B for the committee is the same as picking persons B and A. If we count the two people in both orders (LE 10(a)), there are $4 \cdot 3 = 12$ possibilities, but half of them are repetitions so there are only $\frac{4 \cdot 3}{2!} = \frac{4 \cdot 3}{2} = 6$ possible committees.

AB	AC	AD	BC	BD	CD	6 committees
BA	CA	DA	CB	DB	DC	

(b) This is the same situation as LE 10(b), but there won't be as many possibilities because picking persons A, B, and C for the committee is the same no matter what order in which we select them. If we count the three people in all possible orders (LE 10(b)), there are $4 \cdot 3 \cdot 2 = 24$ possibilities, but only one-sixth of them are different committees so there are only $\frac{4 \cdot 3 \cdot 2}{3!} = \frac{4 \cdot 3 \cdot 2}{6} = 4$ possible committees.

ABC	ABD	ACD	BCD	4 committees
ACB	ADB	ADC	BDC	
BAC	BAD	CAD	CBD	
BCA	BDA	CDA	CDB	
CAB	DAB	DAC	DBC	
CBA	DBA	DCA	DCB	

■

Example 5 illustrates a combination. A **combination** is an arrangement of people or objects in which the order does not make a difference. In the example,

$$_4C_2 = \frac{_4P_2}{2!} = 6 \text{ and } _4C_3 = \frac{_4P_3}{3!} = 4$$

The formula for combinations is as follows.

Combination

The number of ways of selecting a subset of r objects (order does not matter) from a set of n objects is

$$_nC_r = \frac{_nP_r}{r!}$$

Apply this formula to the following exercise, noting that the order of the arrangements does not matter.

LE 11 Skill

(a) A school has 30 teachers. In how many ways can the principal choose 3 people to attend a national meeting?

(b) Find out the easiest way to solve this problem with your calculator.

As you can tell, permutations and combinations are quite similar. Both involve choosing r objects from n objects.

Example 6 A college wants to hire a mathematics tutor, a calculus teacher, and a statistics teacher from a group of 10 applicants, each of whom is qualified for all of the jobs. In how many ways can the college fill the 3 positions?

Solution

We are selecting 3 from a group of 10, but is this example a permutation or a combination? The key question is: Does the order (of a choice) matter? In this case, does it matter if A tutors math, B teaches calculus, and C teaches statistics or instead if C tutors math, A teaches calculus, and B teaches statistics? The answer is yes, the order does matter. This makes it a permutation.

$$_{10}P_3 = 10 \cdot 9 \cdot 8 = 720 \text{ ways}$$

See if you can distinguish between a permutation and a combination in the following exercise.

LE 12 Skill

In each part, tell whether the problem concerns a permutation or a combination. Then find the answer using the appropriate formula.

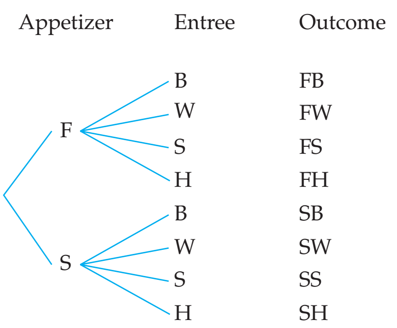
(a) How many different 4-member committees can be formed from 100 U.S. senators?
(b) A meeting is to be addressed by 5 speakers: Al, Baraka, Clyde, Dana, and Elvira. In how many ways can the speakers be ordered?

LE 13 Summary

Tell what you learned about methods of counting in this section. How are permutations related to the Fundamental Counting Principle? What is the difference between a permutation and a combination?

Answers to Selected Lesson Exercises

2. (a)



(b) See the list in part (a).
(c) $2 \cdot 4 = 8$

3. No. Make a list of possibilities.

4. $26^2 \cdot 10^4 = 6{,}760{,}000$

5. $8^2 \cdot 10^8 = 6{,}400{,}000{,}000$

6. $30 \cdot 29 \cdot 28 = 24{,}360$

7. $_{30}P_3 = 30 \cdot 29 \cdot 28 = 24{,}360$

8. (a) $2^6 = 64$ (b) $\frac{6}{64} + \frac{1}{64} = \frac{7}{64}$

9. (a) $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

10. (a) $4 \cdot 3 = 12$ (b) $4 \cdot 3 \cdot 2 = 24$

11. (a) $_{30}C_3 = 4{,}060$

12. (a) Combination; $_{100}C_4 = 3{,}921{,}225$
(b) Permutation; $_5P_5 = 120$

13.3 Homework Exercises

Basic Exercises

1. Jane has 4 skirts and 2 shirts that match.
(a) Draw a tree diagram that shows all possible outfits.
(b) Make an organized list that shows all possible outfits.
(c) How many different outfits can she make?

2. A lottery allows you to select a two-digit number. Each digit may be either 1, 2, or 3. Use a tree diagram to show the sample space and tell how many different numbers can be selected.

3. The Grain Barn is known for its healthy 3-course dinner consisting of appetizer, entree, and vegetable. How many different dinners are possible if you choose 1 item for each course?*

DINNER AT THE GRAIN BARN

Appetizers
Sponge bread
Yogurt gumbo
Buckwheat balls

Entrees
Twice-baked kelp
Oat bran surprise
Flax hash
Marinated bulgur

Vegetables
Scalloped kale
Lima bean puree

4. A car dealer offers the National Motors Nag in 5 colors with a choice of 2 doors or 4 doors and a choice of 2 different engines. How many different possible models are there?*

5. My wife's new bicycle lock is a combination lock with 5 dials, each numbered 1 to 9. If we forget the combination, how many possible combinations are there to try?*

6. In theory, a monkey randomly selecting keys on a typewriter would eventually type some English words. If a monkey is at a keyboard with 48 keys, what is the probability that 5 random keystrokes will produce the word "lucky"?*

7. A fifth-grade class of 26 students wants to elect a president, vice-president, treasurer, and secretary. How many different ways are there to fill the 4 positions?*

8. A club has 15 members. In how many different ways can the club select a president and a vice-president?*

9. If 6 horses are entered in a race and there can be no ties, how many different orders of finish are there?*

10. A department store photographer wants to arrange 3 children in a row for a family photograph. In how many ways can this be done?*

11. (a) Substitute $r = n$ in the formula ${}_nP_r$ and simplify it to obtain the formula for ${}_nP_n$.
(b) Does part (a) involve induction or deduction?

12. Why can't you use the permutation formula in Exercise 2?

13. Until 1995, area codes could not have a first digit of 0 or 1, and the second digit had to be a 0 or a 1. The third digit could be any number.
(a) How many possible area codes were there? (In fact, some codes, such as 411 and 911, are used for other purposes.)
(b) In 1996, there were about 140 different area codes in the United States and Canada. Were the countries close to running out of possible area codes under the old system?
(c) Starting in 1995, the second digit was no longer restricted. How many possible area codes are there now?

*For more practice, go to www.cengage.com/math/sonnabend

14. A state license plate consists of 1 letter followed by 5 digits.
(a) How many different license plates can be made?
(b) If no digit could be repeated, how many possible license plates would there be?
(c) A witness to a crime saw the first letter and first three digits of a license plate correctly as A463 but could not see the last two digits. How many license plates could start with A463?

15. You want to design a format for license plates in a state with 4 million cars. How would you do it? Play it safe and make one that will cover at least 6 million cars.

16. You want to design a format for license plates for a state with 20 million cars. How would you do it? Play it safe and make one that will cover at least 25 million cars.

17. A true/false test has 5 questions. What is the probability of getting at least 4 right by guessing all the answers? Tell how you solved the problem.

18. A true/false test has 4 questions. What is the probability of getting at least 3 right by guessing the answers randomly? Tell how you solved the problem.

19. A quiz has 4 multiple-choice questions. Each question has 5 choices, 1 being correct and the other 4 being incorrect. Suppose you guess on all 4 questions.
(a) How many ways are there to answer the 4 questions?
(b) If someone got at least 3 right, exactly how many questions could he or she have gotten right?
(c) What is P(at least 3 right)?

20. (a) In the preceding exercise, how many ways are there to get no questions right?
(b) What is P(none right)?

21. How many possible 5-card poker hands are there? (*Note:* A regular deck has 52 cards.)*

22. Suppose a test allows a student to pick which questions to answer. In how many different ways can a student choose a set of 8 questions to answer from a group of 10 questions?*

23. (a) When do we use permutations rather than combinations in counting?
(b) Which is usually greater, the number of combinations of a set of objects or the number of permutations?

24. (a) What does it mean in everyday language to form combinations?
(b) What does a "combination" mean in probability?

25. A consumer group plans to select 2 televisions from a shipment of 8 to check the picture quality. In how many ways can they choose 2 televisions?*

26. A baseball manager wants to arrange 9 starting players into a batting order. How many different batting orders are possible?*

27. How many ways are there to rank (as first and second) the 2 greatest U.S. presidents in history from a list of 12 choices?*

28. In a state lottery game, you choose 6 different numbers from the set of counting numbers 1 to 50 (inclusive). The order does not matter. How many different choices can you make?*

Extension Exercises

29. Pascal's triangle is a triangular array of numbers with a special pattern that is useful in some probability and algebra problems. Rows 0–3 of Pascal's triangle are:

Row 0	1
Row 1	1 1
Row 2	1 2 1
Row 3	1 3 3 1

(a) Write Row 4 of Pascal's triangle.
(b) Use a calculator to find the values of ${}_4C_0$, ${}_4C_1$, ${}_4C_2$, ${}_4C_3$, and ${}_4C_4$. What do you notice about the results?
(c) Use Pascal's triangle to find ${}_3C_2$ and ${}_5C_3$. Check your results with a calculator.

*For more practice, go to www.cengage.com/math/sonnabend

30. A quiz has 5 multiple-choice questions. Each question has 4 choices; 1 is the correct answer and the other 3 are incorrect. Suppose that you guess all the answers.

(a) P(at least 1 right) = 1 − P(________).

(b) Use the equation in part (a) to find the probability of getting at least 1 right.

31. A slot machine has the following symbols on its 3 dials. Each dial is spun independently at random and lands on 1 of 20 possibilities.

Dial 1	Dial 2	Dial 3
1 bar 1 bell 2 cherries 7 lemons 8 oranges 1 plum	1 bar 2 bells 2 cherries 1 lemon 6 oranges 8 plums	1 bar 2 bells 1 cherry 7 lemons 4 oranges 5 plums

The payoff for each nickel is as follows.

Payoffs

1 cherry	$.10	3 bells	$5
2 cherries	$.50	3 bars	$10
3 cherries	$1		

(a) How many possible outcomes are there for the 3 dials taken together?

(b) How many ways are there to get 3 cherries (1 on each dial)?

(c) How many ways are there to get 3 bars?

(d) How many ways are there to get 3 bells?

(e) Give the probabilities of the events in parts (b), (c), and (d).

(f) How many ways are there to spin exactly 1 cherry?

(g) How many ways are there to spin exactly 2 cherries?

(h) Give the probabilities of the events in parts (f) and (g).

32. A survey asks you to list your first 3 choices for the Democratic presidential nomination from a list of 6 contenders and your first 3 choices for the Republican presidental nomination from a list of 5 contenders. In how many ways can the survey be completed? (Assume that you choose all of your responses from the lists.)

33. Three couples are sitting together at a show. In how many ways can they sit without any of the couples being split up? Find out by answering the following.

(a) How many choices are there for the aisle seat?

(b) Once the aisle seat is filled, how many choices are there for the seat next to it?

(c) Now how many choices are there for the next seat in?

(d) Now how many choices for the fourth seat?

(e) Now how many choices are there for the fifth seat?

(f) Now how many are there for the sixth seat?

(g) How many ways are there, then, to seat all 6 people?

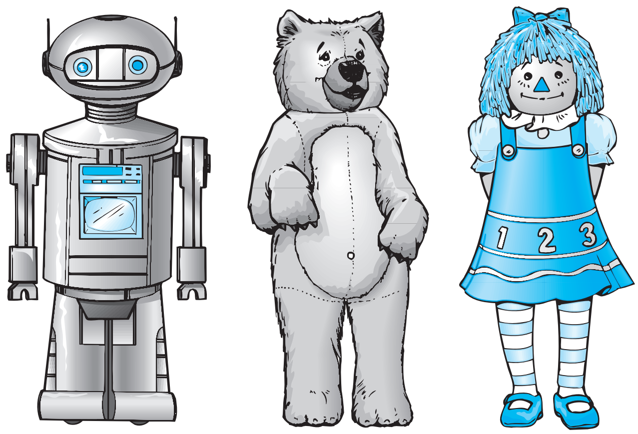
34. A pizza parlor offers the following toppings: mushrooms, peppers, broccoli, shrimp, and pepperoni. How many different kinds of large pizzas (with tomato and cheese) are there? (Include a pizza with no toppings.)

Project

35. Photocopy the 3 characters shown here, and cut each one into 3 separate sections (head, midsection, legs).

(a) Construct some new characters using the pieces.

(b) How many possible characters could you construct?



13.4 Independent and Dependent Events

NCTM Standards

- compute probabilities for simple compound events, using such methods as organized lists, tree diagrams, and area models (6–8)

If you feel sick in the morning, it affects the probability that you will go to school or work. In the study of probability, it is important to know whether the outcome of one event affects the outcome of another. On the other hand, many things that happen have nothing to do with one another.

LE 1 Opener

Consider the following events.

R = rain tomorrow

U = you carry an umbrella tomorrow

H = coin flipped tomorrow lands on heads

(a) Does the probability of R affect the probability of U?
(b) Does the outcome of the coin flip affect whether or not it will rain tomorrow?

In LE 1, R and H are independent events, whereas R and U are dependent events. Two events are **independent** if the probability of one remains the same regardless of how the other turns out. Events that are not independent are **dependent**.

LE 2 Concept

You roll a regular red die and a regular green die. Consider the following events.

A = a 4 on the red die

B = a 1 on the green die

C = a sum of 2 on the two dice

Tell whether each pair of events is independent or dependent.

(a) A and B **(b)** B and C

In LE 2, the outcome of the red die is not affected by the outcome of the green die. The events A and B are totally separate (independent). The probability of B is the same no matter how A turns out. (Similarly, the probability of A is the same no matter how B turns out.)

Classroom Connection

LE 3 Reasoning

Suppose you flip a regular coin 6 times, and it comes up heads every time. Three sixth graders make the following claims. Sam says the next toss is more likely to be heads, because the coin has come up heads so much before. Mike says the next toss is more likely to be tails, because things have to even out. Janet says the chances of heads and tails on the next toss are each 50%.

(a) Who is right?
(b) How would you explain why to the other two students?

Independent Events

What is the probability of two independent events occurring in succession? Try the following exercise.

LE 4 Reasoning

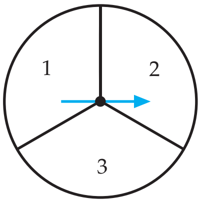


Figure 13–10

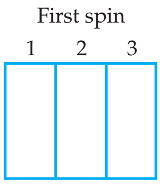


Figure 13–11

Suppose a spinner contains congruent regions numbered 1, 2, and 3 (Figure 13–10).

(a) In an experiment, you spin twice. Are the results of the two spins independent or dependent?
(b) Write a sample space of equally likely outcomes for the experiment. (Use an organized list or tree diagram.)
(c) Suppose A is the event that the first spin is a 1 and B is the event that the second spin is a 2. Compute $P(A)$, $P(B)$, and $P(A \text{ and } B)$.
(d) You can also find $P(A \text{ and } B)$ with an area model. Copy the rectangle in Figure 13–11. To show the result of the second spin after a first spin of 1, divide the column of the rectangle labeled 1 into 3 equal parts. Label or shade the part that would represent a second spin of 2. What fraction of the original rectangle represents a first spin of 1 and a second spin of 2?
(e) Suppose you get a 1 on the first spin $\frac{1}{3}$ of the time. Next, suppose that $\frac{1}{3}$ of those times that you get a 1 on the first spin, you also get a 2 on the second spin. The fraction of the time that you get a 1 on the first spin *and* a 2 on the second spin is $\frac{1}{3}$ of $\frac{1}{3}$ = ________ $\times$ ________ = ________.
(f) What formula relating $P(A)$, $P(B)$, and $P(A \text{ and } B)$ is suggested by parts (c) (d), and (e)?

This spinner exercise is an example of the independent-events formula.

The Independent-Events Formula

If A and B are independent events, $P(A \text{ and } B) = P(A) \cdot P(B)$.

Apply this formula in the following exercise.

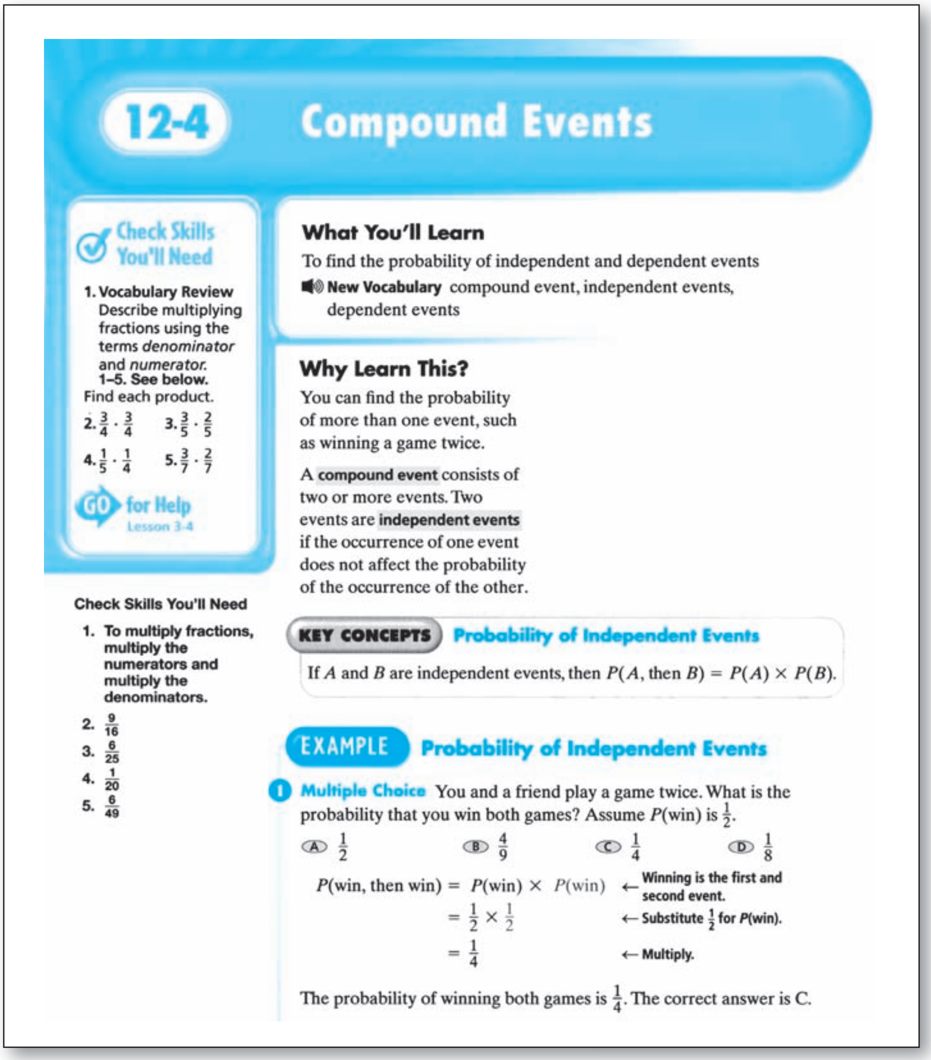
LE 5 Skill

A fair die is tossed twice.

(a) What is the probability of getting a 3 on the first toss followed by an odd number on the second toss?
(b) Draw and shade a rectangular area model that shows the probability in part (a). (*Hint:* Use a 6-by-6 table or a 6-by-2 table.)

You can compute the probability of 3 or more independent events occurring in succession in the same way, by multiplying the probabilities of the individual events. Figure 13–12 shows how a seventh-grade textbook introduces independent events.

Classroom Connection



12-4 Compound Events

Check Skills You'll Need

1. **Vocabulary Review** Describe multiplying fractions using the terms *denominator* and *numerator.* **1–5. See below.**

Find each product.

2. $\frac{3}{4} \cdot \frac{3}{4}$ 3. $\frac{3}{5} \cdot \frac{2}{5}$

4. $\frac{1}{5} \cdot \frac{1}{4}$ 5. $\frac{3}{7} \cdot \frac{2}{7}$

GO for Help Lesson 3-4

Check Skills You'll Need

1. **To multiply fractions, multiply the numerators and multiply the denominators.**
2. $\frac{9}{16}$
3. $\frac{6}{25}$
4. $\frac{1}{20}$
5. $\frac{6}{49}$

What You'll Learn

To find the probability of independent and dependent events

New Vocabulary compound event, independent events, dependent events

Why Learn This?

You can find the probability of more than one event, such as winning a game twice.

A **compound event** consists of two or more events. Two events are **independent events** if the occurrence of one event does not affect the probability of the occurrence of the other.

KEY CONCEPTS **Probability of Independent Events**

If A and B are independent events, then $P(A, \text{then } B) = P(A) \times P(B)$.

EXAMPLE **Probability of Independent Events**

1 **Multiple Choice** You and a friend play a game twice. What is the probability that you win both games? Assume $P(\text{win})$ is $\frac{1}{2}$.

Ⓐ $\frac{1}{2}$ Ⓑ $\frac{4}{9}$ Ⓒ $\frac{1}{4}$ Ⓓ $\frac{1}{8}$

$P(\text{win, then win}) = P(\text{win}) \times P(\text{win})$ ← Winning is the first and second event.

$= \frac{1}{2} \times \frac{1}{2}$ ← Substitute $\frac{1}{2}$ for $P(\text{win})$.

$= \frac{1}{4}$ ← Multiply.

The probability of winning both games is $\frac{1}{4}$. The correct answer is C.

Figure 13–12 Independent events in grade 7

Dependent Events

Now investigate the probability of two dependent events occurring in succession.

MM

LE 6 Reasoning

A box contains cards numbered 1, 2, and 3. You pick 2 cards in succession *without replacement*. ("Without replacement" means you do not put back a card after you pick it.)

1	2	3

(a) In this experiment, are the first and second draws independent or dependent events?

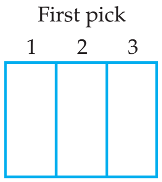


Figure 13–13

(b) Write a sample space of equally likely outcomes for the experiment. (Use an organized list or a tree diagram.)

(c) Suppose *A* is the event that the first card is a 2 and *B* is the event that the second card is a 3. The *P*(*B* given *A*) means the probability of *B*, given that *A* has happened. Compute *P*(*A*), *P*(*B* given *A*), and *P*(*A* and *B*).

(d) You can also find *P*(*A* and *B*) with an area model. Copy the rectangle in Figure 13–13. To show the result of the second pick after a first pick of 2, divide the column of the rectangle labeled 2 into 2 equal parts, and label or shade the part that would represent a second pick of 3. What fraction of the original rectangle represents a first pick of 2 and a second pick of 3?

(e) Suppose *A* occurs $\frac{1}{3}$ of the time. Next, suppose that $\frac{1}{2}$ of those times *A* occurs, *B* also occurs. The fraction of the time that *A* and *B* both occur is $\frac{1}{2}$ of $\frac{1}{3}$ = ________ × ________ = ________.

(f) What formula relating *P*(*A*), *P*(*B* given *A*), and *P*(*A* and *B*) is suggested by parts (c), (d), and (e)?

In LE 6(f), you may have guessed the formula for the Multiplication Rule for Probabilities.

Multiplication Rule for Probabilities

$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$

Use this formula in the next exercise.

LE 7 Reasoning

You have a drawer with 10 black socks and 6 white socks. If you select 2 socks at random, what is the probability of picking 2 white socks? (*Hint:* What is the probability that the first sock will be white?)

What is the connection between the independent-events formula and the Multiplication Rule for Probabilities?

LE 8 Reasoning

The independent-events formula is $P(A \text{ and } B) = P(A) \cdot P(B)$. The Multiplication Rule for Probabilities is $P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$.

(a) What is the difference between the two formulas?

(b) If *A* and *B* are independent events, why would $P(B \text{ given } A) = P(B)$?

The preceding lesson exercise shows that the independent-events formula is a special case of the Multiplication Rule for Probabilities in which *P*(*B* given *A*) is simplified to *P*(*B*) because *A* has no effect on *B*.

Now see whether you can find an experimental and a theoretical probability for the following situation.

LE 9 Reasoning

You have 6 black socks and 4 white socks in a drawer. You pick 2 socks at random.

(a) Without computing probabilities, guess the probability of picking a matching pair.
(b) Simulate the experiment 20 times, using 2 colors of chips and a bag. What is your experimental probability for matching a pair of socks?
(c) Devise a plan and solve the following problem. Compute the theoretical probability of picking a matching pair.

LE 10 Summary

Tell what you learned about independent and dependent events in this section. Give an example of each. What is the difference between them?

Answers to Selected Lesson Exercises

1. (a) Yes (b) No

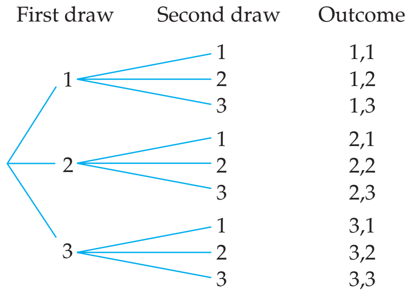
2. (a) Independent
(b) Dependent, because 1 on the green die makes it more likely that you will get a sum of 2.

3. (a) Janet
(b) The coin has no memory of what happened on previous tosses (independent events). The probabilities for each toss are the same.

4. (a) Independent
(b)

1,1	1,2	1,3
2,1	2,2	2,3
3,1	3,2	3,3

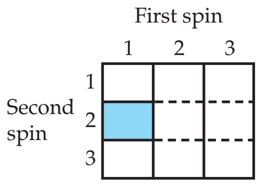
Organized list



Tree diagram

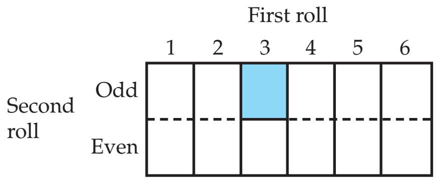
(c) $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{3}$, $P(A \text{ and } B) = \frac{1}{9}$

(d) $\frac{1}{9}$

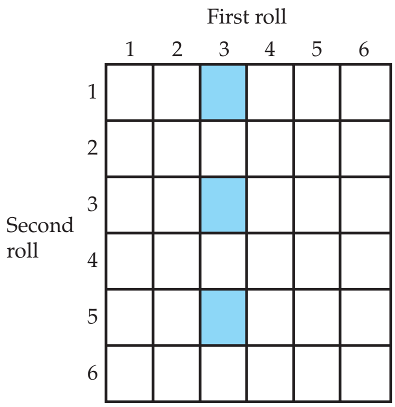


(e) $\frac{1}{3}$; $\frac{1}{3}$; $\frac{1}{9}$

5. (a) $\frac{1}{6} \cdot \frac{3}{6} = \frac{1}{12}$
(b) $\frac{1}{12}$



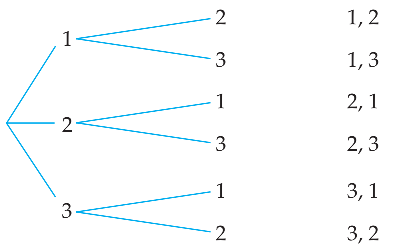
or



6. (a) Dependent

(b)

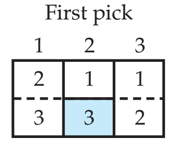
1, 2 1, 3
2, 1 2, 3
3, 1 3, 2



Organized list **Tree diagram**

(c) $P(A) = \frac{1}{3}$, $P(B \text{ given } A) = \frac{1}{2}$, $P(A \text{ and } B) = \frac{1}{6}$

(d) $\frac{1}{6}$



(Second pick shown in each square)

(e) $\frac{1}{2}$; $\frac{1}{3}$; $\frac{1}{6}$

7. $\frac{6}{16} \cdot \frac{5}{15} = \frac{1}{8}$

8. Answers follow the exercise.

9. (c) $\frac{4}{10} \cdot \frac{3}{9} + \frac{6}{10} \cdot \frac{5}{9} = \frac{7}{15}$

13.4 Homework Exercises

Basic Exercises

1. Consider the following events.

C = you eat cereal tomorrow
B = you wear blue shoes tomorrow
M = you drink milk tomorrow

(a) Are C and M independent or dependent events?
(b) Are M and B independent or dependent events?

2. A fair coin is tossed twice.

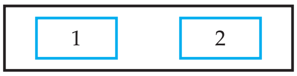
A = heads on the first toss
B = heads on the second toss
C = heads on both tosses

Tell whether each pair of events is independent or dependent.
(a) A and B (b) A and C (c) B and C

3. Suppose A = rolling a sum of 7 with two regular dice. Make up an event B so that
(a) A and B are independent.
(b) A and B are dependent.

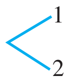
4. Write a sentence that tells the difference between two independent events and two dependent events.

5. A box contains cards numbered 1 and 2. You pick 2 cards in succession with replacement. ("With replacement" means that the number selected first is replaced in the box and the cards are shuffled before the second draw is made.)



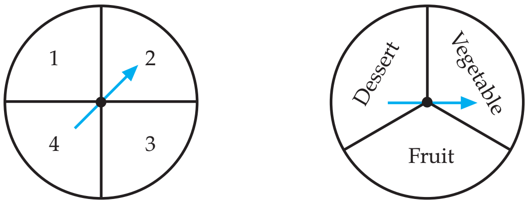
(a) Are the first and second draws independent or dependent events?
(b) Write a sample space of equally likely outcomes.
(c) Complete the tree diagram, showing all the possible outcomes.

First draw **Second draw** **Outcome**



(d) Suppose A = 1 on the first pick and B = 2 on the second pick. $P(A \text{ and } B)$ = ________.
(e) Show how to determine the result of part (d) with a rectangular area model.
(f) Show how to solve part (d) with a formula.

6. A game requires that you spin two different spinners.



You win the number of food items that corresponds to what you spin.

(a) Write a sample space of equally likely outcomes for the experiment. (Use an organized list or a tree diagram.)

(b) What is the probability you will win 4 desserts?

(c) Show how to determine the result of part (b) with a rectangular area model.

(d) Show how to solve part (b) with a formula.

7. Two dice are rolled. What is the probability that both dice show a 5 or a 6? Draw a square area model to support your answer.

8. Consider the following events.

A = college soccer team wins
B = college football team wins

Suppose the college soccer team wins $\frac{1}{2}$ of its games and the college football team wins $\frac{1}{3}$ of its games. What fraction of the time would you expect both teams to win when they play on the same day?

Draw a rectangular area model to support your answer.

9. A man has 2 cars, a Recall and a Sea Bass Brougham. The probability that the Recall starts is 0.1. The probability that the Sea Bass Brougham starts is 0.7. (Assume that the cars operate independently of each other.)

(a) What is the probability that both cars start?

(b) What is the probability that neither car starts?

(c) What is the probability that exactly 1 car starts?

10. Some drug tests are about 98% reliable. This means that there is a probability of 0.98 that the test will correctly identify a drug user or a nonuser. To be safe, each person is tested twice.

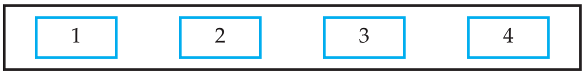
(a) What is the probability that a drug user will pass both tests?

(b) What is the probability that a drug user will fail at least 1 of the tests?

11. A sixth grader says that the probability of rolling two consecutive 1s on a die is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. Is this correct? If not, how would you help the student understand the correct solution?

12. At Risk High, the football coach also teaches probability. He knows that when the opposition calls the coin flip, they call heads 80% of the time. Because heads comes up only 50% of the time, he always lets the visiting team call the coin flip. Should he expect to come out ahead this way?

13. A box contains cards numbered 1, 2, 3, and 4. You pick 2 cards in succession without replacement.



(a) In this experiment, are the first and second draws independent or dependent?

(b) Write a sample space of equally likely outcomes. (Use an organized list or a tree diagram.)

(c) Suppose $A = 3$ on the first pick and $B = 2$ on the second pick. Find $P(A)$, $P(B \text{ given } A)$, and $P(A \text{ and } B)$.

(d) Show how to find $P(A \text{ and } B)$ with an area model.

(e) Show how to find $P(A \text{ and } B)$ using a formula.

14. You pick two cards in succession without replacement from a set of cards numbered 1 through 8 (inclusive).

(a) Are the first and second draws independent or dependent?

(b) Use an area model to determine the probability of drawing a 3 followed by an even number.

(c) Show how to solve part (b) with a formula.

15. You pick 2 cards from a regular deck *without* replacement. What is the probability of picking 2 aces?*

16. A jar contains 7 orange candies and 3 cherry candies. If you pick two at random, what is the probability that you will pick two of those orange candies that you prefer?*

17. (a) When drawing at random with replacement, are the draws independent or dependent?

(b) Without replacement, are the draws independent or dependent?

*For more practice, go to www.cengage.com/math/sonnabend

18. Which topic is easier, independent events or dependent events? Tell why.

19. A drawer contains a mixture of 5 black socks and 8 white socks. You randomly select two socks to wear.
(a) Determine the probability that both socks are black.
(b) What is the probability that both socks match?*

20. A drawer contains a mixture of 10 black socks, 8 white socks, and 4 red socks. You randomly select 2 socks to wear. Determine the probability that
(a) both are black.
(b) both are white.
(c) you pick a matching pair.*

21. Assume that for a nuclear power plant to have an accident, 4 systems must fail. The probabilities that the 4 individual systems fail are 0.01, 0.006, 0.002, and 0.002, respectively. What is the probability that the plant will have an accident?

22. What is the probability that a family with 8 children has 8 girls?

23. A group of mothers and their grown daughters were surveyed.

A = mother attended college
B = daughter attended college

$P(A) = 0.3$, $P(B) = 0.6$, and $P(B \text{ given } A) = 0.7$. What is the probability that a mother and her daughter both attended college?

24. On the average, it rains or snows about 114 days a year in Spokane, Washington.
(a) If you visit for a day, what is the probability that there will be precipitation (rain or snow)?
(b) If there is precipitation one day, the probability of precipitation the next day is 0.5. If you visit for 2 days, what is the probability that it will rain or snow both days?

25. A basketball team has a probability of 0.7 of making each free throw. If they shoot 5 free throws at the end of a game, determine the probability that they make
(a) all 5 shots. (b) at least 4 shots.

26. A slot machine has 3 independent dials set up as follows.

Dial 1	Dial 2	Dial 3
1 bar	1 bar	1 bar
1 bell	2 bells	2 bells
2 cherries	2 cherries	1 cherry
7 lemons	1 lemon	7 lemons
8 oranges	6 oranges	4 oranges
1 plum	8 plums	5 plums

What is the probability of getting each of the following?
(a) 3 lemons (b) 0 cherries

27. A shipment of radios contains 97 that work and 3 that are defective. An inspector selects 5 at random. What is the probability that they all work?

28. How strange are coincidences? Suppose an event has a 1 in 500 chance of happening each day. Won't you be surprised if it occurs? But approximately what is the probability that this event will happen sometime in the next year? (*Hint:* Assume independence, and find the probability that it will not occur in the next year.)

29. The home team is behind by one point. There is one second on the clock as Ben steps up to the free-throw line to shoot "one on one." This means that he takes a one-point free throw (shot). He has to make the free throw to be allowed to take a second one. Ben makes about 60% of his free throws. Solve each of the following with a formula and a decimal square area model (Activity Card 3).
(a) What is the probability that Ben makes 0 free throws (and the team loses)?
(b) What is the probability that Ben makes 2 free throws (and the team wins)?
(c) What is the probability that Ben makes 1 free throw (leading to an overtime game)?

30. When it snows, the local schools are closed about 70% of the time. The forecast for tomorrow says there is a 40% chance of snow. Solve the following with a formula and a decimal square area model (Activity Card 3).
(a) What is the probability that the local schools will close?
(b) What is the probability it will snow, and the local schools will *not* close?

*For more practice, go to www.cengage.com/math/sonnabend

Extension Exercises

31. Suppose tulip bulbs have a probability of 0.6 of flowering. How many bulbs should you buy to have a probability of 0.9 of obtaining at least 1 flower?

32. In a mathematics class that contains 50% boys and 50% girls, 60% of the students are 8 years old and 40% are 9 years old.
(a) What is the largest possible percentage of 9-year-old girls in the class?
(b) What is the smallest possible percentage of 9-year-old girls in the class?
(c) If age and sex are independent, then ________% of the class is 9-year-old girls.

33. Two basketball teams play a best-2-out-of-3 series, meaning that the first team to win two games is the champion. Suppose that team A has a probability of 0.6 of winning each game. Find the probability that team A wins
(a) in 2 games. (b) in 3 games.
(*Hint:* Draw a tree diagram.)

34. A World Series is a best-4-out-of-7 series. If the American League team has a probability of 0.5 of winning each game, determine the probability that they win the World Series in
(a) 4 games. (b) 5 games.

35. Consider how $P(A \text{ and } B)$ is related to $P(A)$.
Let A = Mike will take math next semester.
If possible, make up an event B so that
(a) $P(A \text{ and } B) < P(A)$
(b) $P(A \text{ and } B) = P(A)$
(c) $P(A \text{ and } B) > P(A)$

36. If possible, make up two events A and B that are
(a) independent and not mutually exclusive.
(b) independent and mutually exclusive.
(c) dependent and mutually exclusive.

13.5 Expected Value and Odds

NCTM Standards

- propose and justify conclusions and predictions that are based on data and design studies to further investigate the conclusions or predictions (3–5)
- recognize and apply mathematics in contexts outside of mathematics (pre-K–12)

Have you ever played a lottery or gambled at a casino? Do you know how to compute how much you are going to *lose* in the long run? Do you know what the term "odds" means? This lesson introduces the mathematics of the expected value and odds.

In a casino, people risk their money and usually end up losing some of it. Although individual outcomes vary quite a bit, casinos consistently come out ahead at their games.

Many people are not interested in gambling, but the same mathematics applies to another area that does affect nearly everyone—insurance. In buying insurance, we also risk our money and usually end up losing some.

Fair versus Unfair Games

Would you play a gambling game in which you roll 2 dice and you win the amount you bet whenever the sum is greater than 10? This is an example of an unfair game.

In a **fair game** involving money, a player can expect to come out even in the long run. In a fair game between two players, neither player has an advantage due to the rules.

Discussion

LE 1 Opener

Consider the following game. Place four chips in a cup: two red and two blue. Each player selects one chip from the cup without looking. Player 1 wins if both chips are the same color. Player 2 wins if the two chips are different colors.

(a) Do you think this is a fair game? If not, which player do you think has the advantage?
(b) If you have the game materials, play the game 8 times and tally the results. If not, skip to part (d).
(c) Answer part (a) again.
(d) Use a sample space, tree diagram, probability formulas, or area model to determine the theoretical probability of each player winning.

LE 2 Reasoning

Make up a fair game for 2 players that involves rolling 2 dice.

Expected Value

Casino games are not fair games. If they were, casino owners would not make any money. Consider the following model of a casino game.

LE 3 Concept

In a game, you roll a regular dice. If you roll a 3, the payoff is \$3. If you roll any other number, you are not paid anything.

(a) Suppose you played this game 6 times and obtained perfectly average results. What is the total amount of money you would be paid?
(b) What is your average payoff per game?
(c) The probability of a 3 is $\frac{1}{6}$, so on average, you would receive a \$3 payoff $\frac{1}{6}$ of the time. How can you compute the average payoff from these numbers?

In LE 3(c), you could multiply the payoff times its probability, $\$3 \cdot \frac{1}{6} = \$0.50$, to obtain the average payoff.

LE 4 Concept

Now suppose that a second prize is added to the game in LE 3. If you roll a 3, you receive a payoff of \$3, and if you roll a 6, you receive a payoff of \$6.

(a) What is your average payoff per game?
(b) Fill in the following table and tell how you could compute the average payoff using \$3, \$6, and their respective probabilities.

Payoff	Probability
\$3	
\$6	

(c) If there is a charge of \$2 to play this game, what is the average loss per game in the long run?

The first state lottery, held in New Hampshire in 1964, created a lot of controversy. Now about 41 states use lotteries to raise money for state programs, as an alternative to raising taxes. Is a gambling game a good way to raise money? Many people in the states without lotteries don't think so.

State officials in lottery states need to know how much money they can expect to raise in the long run—the expected value. People playing the lottery are also interested in the expected value of their lottery tickets (Figure 13–14).

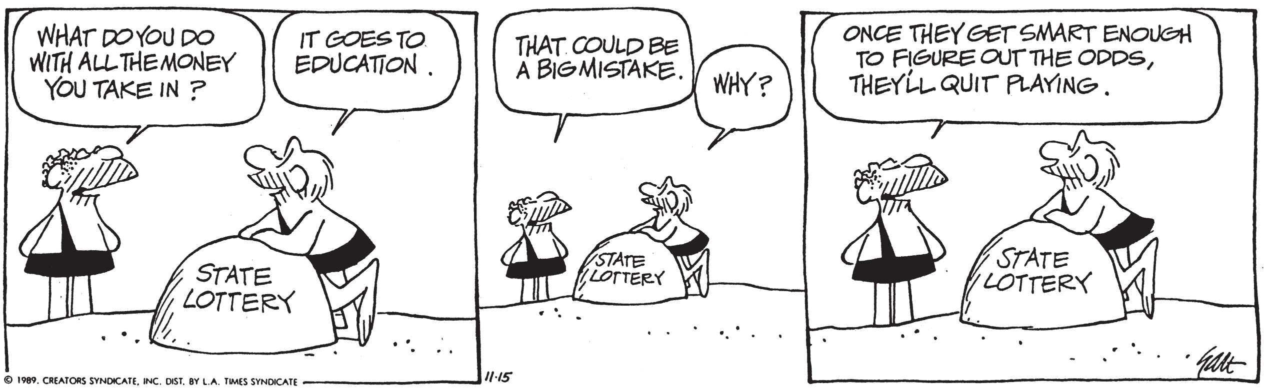


Figure 13–14

The formula used in LE 4(b) is as follows.

Definition: Expected Value

If an experiment has the possible numerical outcomes $n_1, n_2, n_3, \ldots, n_r$ with corresponding probabilities $p_1, p_2, p_3, \ldots, p_r$, then the **expected value** E of the experiment is

$$E = n_1p_1 + n_2p_2 + n_3p_3 + \ldots + n_rp_r$$

Use the expected-value formula in the following exercise.

LE 5 Reasoning

Suppose a lottery game allows you to select a 2-digit number. Each digit may be either 1, 2, 3, 4, or 5. If you pick the winning number, you win $10. Otherwise, you win nothing.

(a) What is the probability that you will pick the winning number?

(b) The notation E(payoff) means the expected (or average) payoff. What is E(payoff)?

(c) If the lottery ticket costs $1, how much should you expect to lose on the average (per play)?

Insurance companies also use expected values. To determine their rates and payoffs, they estimate the probabilities of particular catastrophes.

Example 2 An insurance company will insure your dorm room against theft for a semester. Suppose the value of your possessions is \$800. The probability of your being robbed of \$400 worth of goods during a semester is $\frac{1}{100}$, and the probability of your being robbed of \$800 worth of goods is $\frac{1}{400}$. Assume that these are the only possible kinds of robberies. How much should the insurance company charge people like you to cover the money they pay out and to make an additional \$20 profit per person on the average?

Solution

Understanding the Problem The insurance company wants \$20 per person to be left over after paying out all the claims.

Devising a Plan Compute the expected payout for the insurance. Then, to make \$20 profit, charge the expected payout plus an additional \$20.

Carrying Out the Plan

Payout	Probability
\$400	$\frac{1}{100}$
\$800	$\frac{1}{400}$

$$E(\text{payout}) = (\$400) \cdot \frac{1}{100} + (\$800) \cdot \frac{1}{400} = \$4 + \$2 = \$6$$

They should charge \$6 + \$20 = \$26 for the policy to make an average gain of \$20 per policy.

Looking Back The answer seems reasonable. ■

Discussion

LE 6 Reasoning

Consider the following problem. "In a carnival game, you toss a single die. If you roll a 3, you win \$3. If you roll a 6, you win \$6. Otherwise, you win nothing. How much should the carnival operators charge you to have an expected gain of \$0.50 per game?" Devise a plan, and solve the problem.

Odds

In gambling games, probabilities are often stated using odds. Odds compare your chances of losing and winning on each play. For example, on the race card in the following table, the *odds against* Turf Tortoise winning are 12–1 (12 to 1). This expression means that Turf Tortoise is expected to lose about 12 times for every 1 time he wins. Whereas a probability compares your chance of losing or winning to the *total number of possibilities*, odds compare your possibilities of losing and winning to *each other*.

Post Time 1 P.M.

Race 1: 6 Furlongs 3YO; CLM $5000

Hot Air	6–5
Dog Lover	3–1
Wet Blanket	4–1
Lost Marbles	5–1
Lead Balloon	7–1
Turf Tortoise	12–1
Obstacle	20–1

Actually, the real chance of Turf Tortoise winning is slightly worse than 12 to 1. The track pays off bettors based upon the odds *after* taking out about 20% of the money for expenses. For this reason, the track adjusts the odds by a factor of about 20%.

Odds (against) can be defined as follows.

Definition: Odds of Experiments with Equally Likely Outcomes

Odds against = number of unfavorable outcomes to number of favorable outcomes

Odds in favor = number of favorable outcomes to number of unfavorable outcomes

LE 7 Concept

(a) In the table for Race 1, the odds against Wet Blanket's winning are 4–1 (4 to 1). What does this statement mean?

(b) What are the odds in favor of Wet Blanket winning?

Probabilities describe the frequency of a (favorable) result in relation to all possible outcomes. "Odds against" compare unfavorable and favorable results, such as losing and winning. Any probability can be converted to odds, and any odds can be converted to a probability.

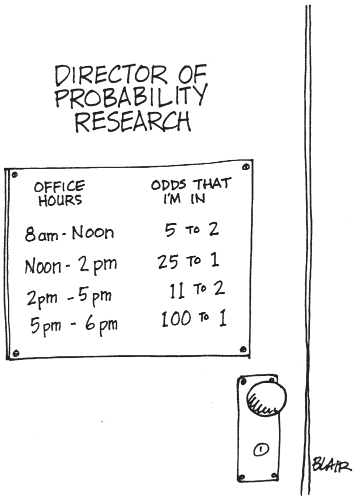


Figure 13–15

Example 3

What is the probability that the director whose door is shown in Figure 13–15 is in his office at 1 P.M.? (Assume the odds are against.)

Solution

Odds of "25 to 1 against" mean that the director will not be there 25 times for each 1 time the director is there. The probability of the director's being there is 1 out of 26, or $\frac{1}{26}$. ■

The preceding example illustrates the relationship between the odds against E and the probability of E. You can compute the odds against E using the formula $\frac{P(\overline{E})}{P(E)}$.

LE 8 Skill

The odds against Dead Weight winning the Crabgrass Derby today are 7 to 1. Express the probability of winning using a fraction, a decimal, and a percent.

LE 9 Skill

A roulette wheel has 38 numbers. If I bet on 1 number, what are the odds against my winning?

The odds are the basis for payoffs on bets. If you were paid off on the basis of the *true* odds, the game would be fair. You would tend to come out even in the long run. For example, in LE 9, suppose you gain $37 for each $1 bet (from odds of 37 to 1) when you picked the winning number. The expected value could be computed from the following chart.

Gain	Probability
37	$\frac{1}{38}$
−$1	$\frac{37}{38}$

$$E(\text{Gain}) = \$37\left(\frac{1}{38}\right) + (-\$1)\left(\frac{37}{38}\right) = \$0$$

When the expected value is $0, it is a fair bet, because you would not expect to gain or lose money in the long run.

Because casinos want to make money, they pay out the money at a slightly lower rate. If you made this bet in roulette, you would actually gain $35, rather than $37, on a $1 bet.

LE 10 Summary

Tell what you learned about fair games and expected value in this section. How is the expected value like the mean of a data set?

Answers to Selected Lesson Exercises

1. (d) Player 2 has a probability of $\frac{2}{3}$ of winning each game.

R_1R_2 R_1B_1 R_1B_2 R_2B_1 R_2B_2 B_1B_2
R_2R_1 B_1R_1 B_2R_1 B_1R_2 B_2R_2 B_2B_1

3. (a) $3 (win once)
(c) Answer follows the exercise.

4. (a) $9 payoff for 6 games; $1.50 per game
(b) Probabilities are both $\frac{1}{6}$
(c) $2 − $1.50 = $0.50 loss per game

5. (a) $\frac{1}{25}$ (b) $0.40 (c) $0.60

6. $E(\text{Payoff}) = \frac{1}{6}(\$3) + \frac{1}{6}(\$6) = \$1.50$
$0.50 + $1.50 = $2

7. (a) If the odds are accurate, Wet Blanket can be expected to lose 4 times for every 1 time he wins under these conditions.
(b) 1 to 4

8. $\frac{1}{8}$, 0.125, 12.5%

9. 37 to 1

13.5 Homework Exercises

Basic Exercises

1. In a dice game, you roll 2 dice. If the sum is divisible by 3 or 5, you win. Otherwise, your opponent wins. Whom does this game favor?

2. Make up a fair game for 2 players that involves flipping 2 coins.

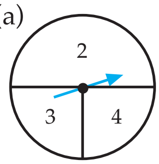
3. Consider the following game. Place four chips in a cup: three of one color and one of another color. Each player selects one chip from the cup without looking. Player 1 wins if both chips are the same color. Player 2 wins if the two chips are different colors.
(a) Do you think this is a fair game? If not, which player do you think has the advantage?
(b) If you have the game materials, play the game 9 times and tally the results. If not, skip to part (d).
(c) Answer part (a) again.
(d) Use a sample space, tree diagram, or area model to determine the theoretical probability of each player winning.

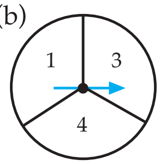
4. Someone offers to play a dice game called "odd-even" with you. In this game, you roll two dice. If the product is odd, you win. If the product is even, your opponent wins. Is this a fair game? If not, whom does it favor?

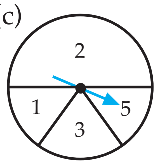
5. In a lottery game, you have a $\frac{1}{10}$ chance of winning \$1,000 and a $\frac{1}{10}$ chance of winning \$500. What is the expected (average) value of the payoff for a single game?

6. You have a job working for a mathematician. She pays you each day according to what card you select from a bag. Two of the cards say \$200, five of them say \$100, and three of them say \$50. What is your expected (average) daily pay?

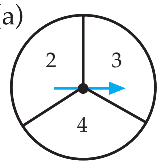
7. In a game, the number you spin is the number of dollars you win. Which of the following spinners would be the best one to spin?

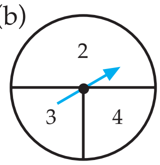
(a)


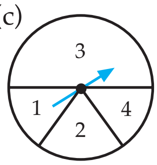
(b)


(c)


8. In a game, the number you spin is the number of dollars you win. Which of the following spinners would be the best one to spin?

(a)


(b)


(c)


9. (a) The Unshakeable Insurance Company will insure your dorm room against theft. The insurance costs \$25 per year. If the probability of getting robbed of an average of \$500 worth of possessions is $\frac{1}{100}$, how much profit does the insurance company expect to make from your \$25 insurance premium?
(b) The company sells \$100,000 worth of automobile liability insurance that costs \$50 per year. In the past, about 1 out of every 1,000 people have collected on a claim. The average (mean) claim is \$30,000. What is the expected payoff on your \$50 investment?
(c) Why might the insurance in part (b) be worth the investment, whereas the insurance in part (a) is probably not worth it?

10. A particular oil well costs \$40,000 to drill. The probability of striking oil is $\frac{1}{20}$.
(a) If an oil strike is worth \$500,000, what is the expected gain from drilling?
(b) Would you go ahead and drill?

11. An apartment complex has 20 air conditioners. Each summer, a certain number of them have to be replaced.

Number of Air Conditioners Replaced	Probability
0	0.21
1	0.32
2	0.18
3	0.11
4	0.11
5	0.07

What is the expected number of air conditioners that will be replaced in the summer?

12. The probability that a particular operation is successful is 0.32.
(a) If the operation is performed 600 times a year at a hospital, about how many successful operations can the hospital expect?
(b) If you decide to have this operation, give some reasons why the probability of its success for you might be somewhat higher or lower than 0.32.

13. Madame Wamma Jamma will predict the sex of a future baby for $5. If she is wrong, she even has a money-back guarantee! If Madame Wamma Jamma has no psychic power (Perish the thought!), what is her expected gain per customer? Assume that she cheerfully refunds the $5 whenever she is wrong.

14. (a) Suppose you buy $50,000 of air travel insurance against death. If the probability of your dying on the flight is 1 in 300,000, what is the mean payoff for your insurance policy?
(b) If the insurance costs $1, what is the average gain or loss per policy for consumers?

15. State lotteries sell about $20 billion worth of tickets per year. In Vermont, you choose a 3-digit number. If it matches the state's number, the state pays you $500.
(a) How many 3-digit numbers are there?
(b) What is your chance of picking the winning number?
(c) What is the average payoff you would receive in the long run?
(d) Tickets cost $1. How much can you expect to lose per play?

16. A lottery game allows you to select a 3-digit number using the digits 0, 1, 2, and 3. You may use the same digit more than once. If you pick the winning number, you are paid $25 on a $1 bet.
(a) What is the average payoff?
(b) How much would you expect to lose, on the average, per $1 bet?

17. Recall that a roulette wheel has 38 slots. Each slot is the same size. Eighteen are red, 18 are black, and 2 are green.
(a) What is the probability of black?
(b) If you bet $1 on black, you receive a $2 payoff if the result is black and $0 if the result is red or green. What is the average loss on a $1 bet on black?
(c) You can also bet on 6 different numbers. If any of them comes up, you receive $6 back for each $1 bet. What is the expected gain or loss on a $1 bet?

*For more practice, go to www.cengage.com/math/sonnabend

18. You are playing "Deal or No Deal." If you continue, you will randomly select one of the remaining five briefcases that contain $5, $100, $50,000, $100,000, and $300,000, respectively. The game show host offers you $72,000 to quit playing at this point.
(a) What is the expected value of the briefcase you would select from the five that remain?
(b) Would you accept the $72,000 offer?

19. In a gambling game, you receive a payoff of $10 if you roll a sum of 10 and $7 if you roll a sum of 7 on two dice. Otherwise, you receive no payoff. What is the average payoff per play?*

20. In a carnival game, you roll 2 dice. If the sum is 5, you receive a $5 payoff. If the sum is 10, you receive a $10 payoff. Otherwise, you receive no payoff. What is the expected payoff?*

21. In a gambling game, you pick 1 card from a standard deck. If you pick an ace, you are paid $10. If you pick a picture card (J, Q, or K), you are paid $5. Otherwise, you win nothing. How much should a carnival booth charge you to play this game if they want an average profit of $0.40 per game?

22. You are designing a spinner game for a carnival. You want to charge people $1 and estimate that 500 people will play. You would like to make about $100. Sketch a spinner, and give the rules and payoffs.

23. If you purchase a handgun to keep at home, the odds are 11 to 1 against using the gun on an intruder rather than on yourself, a family member, or someone you know. This means that about ____________ of every ____________ gun owners who use handguns will use it on an intruder.

24. Are you ready for a cheery problem? The following table gives the odds against dying in the next year.

Age	Odds Against Dying in the Next Year: *Male*	*Female*
15-24	576–1	1814–1
25-34	560–1	1489–1
35-44	384–1	743–1
45-54	140–1	269–1
55-64	57–1	111–1
65-74	25–1	49–1
75-84	11–1	17–1
85-up	6–1	7–1

(Continued on the next page)

What is the probability that you will live another year?

25. Data indicate that about 2 out of 5 marriages last at least 25 years.
(a) The odds against a marriage lasting at least 25 years are ____________ .*
(b) What are the odds in favor of a marriage lasting at least 25 years?

26. A roulette wheel has 38 slots, of which 18 are red, 18 black, and 2 green.
(a) If I bet on red, what are the odds against my winning?*
(b) What are the odds in favor of my winning?

Extension Exercises

27. A company wants to test its employees for a drug. The test is 98% accurate. If someone is using the drug, that person tests positive 98% of the time. If someone is not using the drug, that person tests negative 98% of the time.
(a) Suppose the company randomly tests all 10,000 of its employees. Also, assume that 50 people are actually using the drug. How many people who are not using the drug will test positive (false positive)?
(b) In part (a), what is the probability that someone who tests positive is actually using the drug?
(c) Suppose that, instead, the company tests only people who exhibit suspicious behavior. Suppose 100 people act suspiciously, and 30 of them are using the drug. How many people who are not using the drug will test positive (false positive)?
(d) In part (c), what is the probability that someone who tests positive is actually using the drug?

28. A company wants to test its employees for a drug. The test is 95% accurate.
(a) Suppose that the company randomly tests all 5,000 of its employees. Furthermore, assume that 100 people are actually using the drug. How many people who are not using the drug will test positive (false positive)?
(b) In part (a), what is the probability that someone who tests positive is actually using the drug?

(Continued in the next column)

(c) Suppose that, instead, the company tests only people who exhibit suspicious behavior. Suppose 150 people act suspiciously and 80 of them are using the drug. How many people who are not using the drug will test positive (false positive)?
(d) In part (c), what is the probability that someone who tests positive is actually using the drug?

29. A patient has a stroke and must choose between brain surgery and drug therapy. Of 100 people who have the surgery, 20 die during surgery, 15 more die after 1 year, and 5 more die after 3 years. Of 100 people who receive drug therapy, 5 die almost immediately, 20 more die after 1 year, and 30 more die after 3 years. Which treatment would you advise someone to take? Why?

30. Refer to the slot machine in Homework Exercise 31 from Section 13.3. Use the probabilities you found to compute the expected payoff on a $0.05 bet.

31. Consider the following problem: "A football betting service picks 1 game each week. They charge $100 per pick, but only if they pick the game correctly. The service starts with 200 customers. They tell 100 customers to bet on one team and 100 customers to bet on the other team. Assume that this continues for 10 weeks, with the following results. Each week, only the customers who win continue the next week. Furthermore, the service attracts 40 new customers each week. How much can the service expect to make if none of the games are ties?" Devise a plan, and solve the problem.

32. Maria and Dionne have a basketball contest. The first one to make a shot wins. They each make their shots about half of the time. Because Maria brought the basketball, they decide she will shoot first. What is Maria's chance of winning? Show how to solve this with an area model.

Project

33. Go to a website that gives information about state lotteries. Select two different state lotteries. See if you can find the expected payoff for each lottery and tell which would be better to play.

*For more practice, go to www.cengage.com/math/sonnabend

Chapter 13 Summary

Uncertainty is part of our everyday lives. It has been said that nothing in life is certain but death and taxes. For this reason, probability theory is helpful for studying the likelihood of everyday events.

Probabilities tell us approximately what we can expect to happen when the same event is repeated many times under the same conditions. If a sample space of equally likely events can be written for an experiment, it may be possible to compute a theoretical probability.

Experimental probabilities are based on experimental results under identical or similar conditions. If it is impractical or costly to find an experimental probability of a random event, one can sometimes study a simulation of the event, using coins, dice, or a computer.

To decide how many letters and digits to put in license plates, phone numbers, and codes, people use counting techniques. These same techniques indicate how likely one is to guess correctly on a multiple-choice test. In probability, counting techniques are used to determine the sizes of sample spaces.

In computing the probability that events A and B will occur, one multiplies probabilities. If A and B are dependent, one must calculate how much one event affects the probability of the other:

$$P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$$

If events A and B are independent, meaning they do not influence each other's probabilities, one can compute $P(A \text{ and } B)$ simply by multiplying $P(A) \cdot P(B)$.

How do casinos design gambling games, states design lotteries, and insurance companies set fees? They all use probabilities to estimate the expected average payoffs per person. They then use those payoffs to determine a fee that will cover the payoffs and other expenses.

Study Guide

To review Chapter 13, see what you know about each of the following ideas or terms that you have studied. You can also use this list to generate your own questions about the chapter.

Probability in Grades 1–8

The following chart shows at what grade levels selected probability topics typically appear in elementary- and middle-school mathematics textbooks.

Topic	Typical Grade Level in Current Texbooks
Basic probability	3, 4, 5, 6
Experimental and theoretical probability	5, 6, 7, 8
Simulations	6, 7, 8
Fundamental Counting Principle	5, 6, 7
Permutations and combinations	7, 8
Independent events	6, 7
Dependent events	7, 8

Review Exercises

1. A judge rates the best and second-best orange juices out of 4 brands, *A*, *B*, *C*, and *D*. What is the sample space for his pair of choices?

2. (a) What is the probability of rolling a product less than 10 on 2 dice?
(b) Describe how you could determine the same probability experimentally.

3. What is the probability of getting 3 heads and 1 tail when you flip 4 coins?

4. Write a paragraph that defines experimental and theoretical probability and tells how these two kinds of probabilities are related.

5. A pollster asked 300 students which of the following pizza toppings they prefer: mushrooms, pepperoni, or spinach.

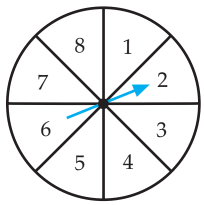
	Mushroom	Pepperoni	Spinach
High school	50	80	25
College	50	60	35

On the basis of these results, what is the probability of each of the following?
(a) A college student prefers pepperoni.
(b) A student prefers spinach.

6. Suppose A = you pass the next math test. Make up an event B so that A and B are
(a) mutually exclusive. (b) not mutually exclusive.

7. A caterer plans to offer people the choice of turkey, chicken, peanut butter, or vegetable sandwiches. On the basis of past orders, the probability someone will choose a chicken sandwich is $\frac{1}{4}$, and the probability someone will choose a turkey sandwich is $\frac{1}{5}$. What is the probability that someone chooses neither chicken nor turkey?

8. You have a spinner like the one shown. You want to simulate each *second* at a one-way traffic intersection. In preliminary work, you found that 150 cars passed in 10 minutes. How would you use the spinner in the simulation?



9. New Hay Checks come with one of 4 different collectible famous-thoroughbred cards in each box. Using a spinner with equal regions numbered 1 to 4, I simulated buying boxes until I had collected all 4 cards. The results follow.

1 3 1 2 4
2 4 1 1 2 2 3
4 3 1 2
4 3 4 1 2
1 3 3 1 4 1 3 2

On the basis of these results, how many boxes would you expect to buy to collect all 4 cards?

10. Milka has 4 shirts and 3 skirts that match. Draw a tree diagram that shows all possible outfits and tell how many there are.

11. A state has 2 kinds of license plates: plates with 2 letters followed by 2 digits, and plates with 2 letters followed by 3 digits. How many different plates can the state make? Tell how you solved the problem.

12. A true/false quiz has 6 questions. If I guess at random, what is the probability that I will get at least 5 right? (Assume that I answer each question either *true* or *false*.) Tell how you solved the problem.

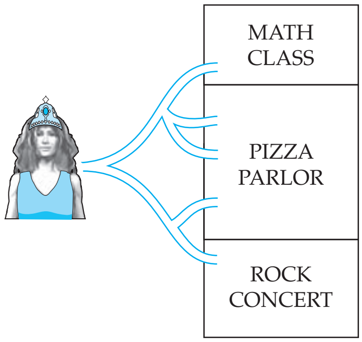
13. A jury of 12 is to be selected from 20 eligible jurors. How many different juries are possible?

14. A seventh grader asks what the difference is between a permutation and a combination. What will you tell the student?

15. A fortune teller claims that the card she selects is related to your future. She would say the card and your future are ____________ events. In fact, they are ____________ events.

16. Suppose A = you pass the next math test. Make up an event B so that A and B are
(a) independent.
(b) dependent.

17. A princess walks through the following maze, choosing her path at random.



Use a rectangular area model to determine the probability that she ends up at
(a) the math class. (b) the pizza parlor
(c) the rock concert.

18. Show how to solve exercise 17 without a rectangular area model.

19. A defective undergarment is inspected by inspector 12 and inspector 14. Each of them has a 0.9 chance of finding the defect. What is the probability that neither one of them will find the defect?

20. A drawer contains a mixture of 8 black socks and 6 white socks. You randomly select two socks to wear.
(a) Determine the probability that both socks are white.
(b) What is the probability that both socks match?

21. A motel chain has found that the probability that a customer pays by check is 0.2. The probability that the check is no good, given that a customer pays by check, is 0.1. What is the probability that
(a) a customer does not pay by check?
(b) a customer pays by check and it is no good?

22. Make up a fair game for 2 players that involves flipping 3 coins.

23. An insurance company will insure your home against theft. The value of your possessions that are insurable is $1,000. Suppose the probability of your being burglarized of $500 worth of goods is $\frac{1}{200}$, and the probability of your being burglarized of $1,000 worth of goods is $\frac{1}{1000}$. Assume that these are the only kinds of burglaries possible. How much should the insurance company charge people like you to make an average profit of $10 per policy?

24. A dice game pays $3 back on a $1 bet if you roll a sum of 7 on two dice and $10 back on a $1 bet if you roll a sum of 11. Otherwise, you lose the $1.
(a) What is E(payoff on $1 bet)?
(b) What is E(gain on $1 bet)?

25.

Courtesy of Library of Congress

Sure Thing	1–2
Hopeless	4–1
Bad Breath	6–1
Unsettled	7–1
Lethargic	10–1
Why Bother	50–1

(a) According to the odds against each horse, Unsettled (see photo) wins about 1 out of _____ races.
(b) The probability of Lethargic winning is about _____.
(c) What are the odds in favor of Bad Breath winning?

Alternate Assessment

Do one of the following assessment activities: add to your portfolio, add to your journal, write another unit test, do another self-assessment, or give a presentation.