Mathematical Logic & Dirichlet's Principle

Session 3

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Albion Online Intellectual Club

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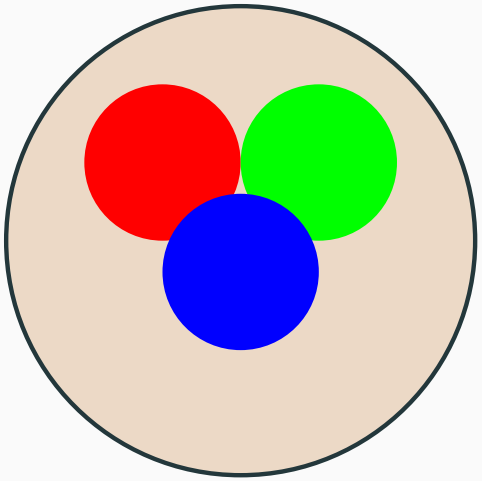
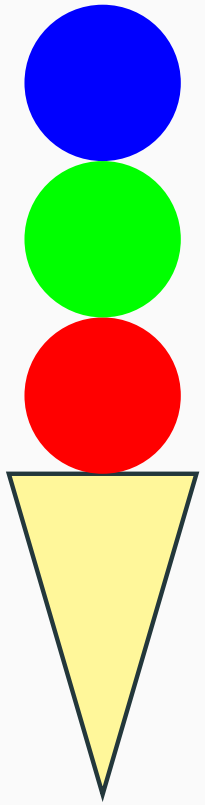
Homework Solutions

Homework Problem 1: Combinatorics - Ice Cream Flavours (Permutations and Combinations)

Problem Statement: An ice cream shop offers 10 different flavours of ice cream. You choose a bowl that contains 3 different flavours. How many unique combinations of 3 flavours can you select for the bowl?



Ice Cream Bowl



Ice Cream Cone

Homework Problem 1: Combinatorics - Ice Cream Flavours (Argument)

1. For your first flavour, you can choose any one of the 10 flavours.
2. For the second flavour, you can choose from the remaining 9 flavours (because you already picked one).
3. For the third flavour, you can choose from the remaining 8 flavours.

So, the total number of ways to pick the flavours would be:

$$10 \times 9 \times 8 = 720$$

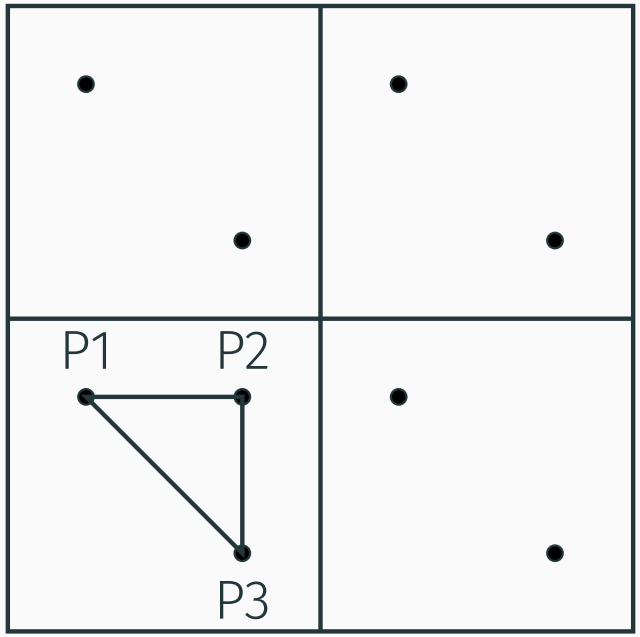
Now, we divide 720 by 6 to get the correct number of combinations (to remove repeats):

$$\frac{720}{6} = 120$$

Homework Problem 2: Points in a square

Problem Statement: Consider a square with side length 1. Prove that if nine points are placed inside the square, there will always be a set of three points that form a triangle with an area of 1/8 or less.

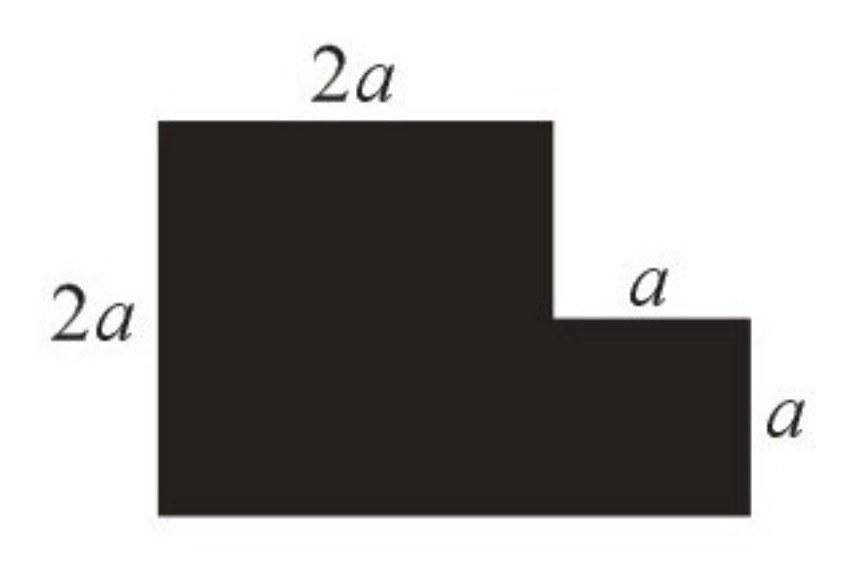
Homework Problem 2: Visualisation



Homework Problem 3: Creating new geometry

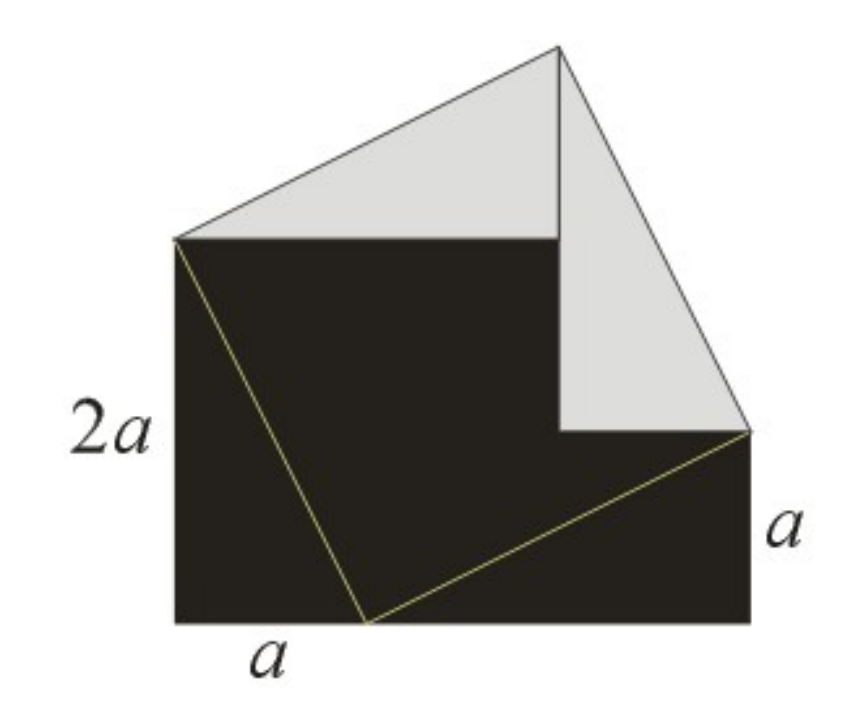
Problem Statement:

Cut the figure into three pieces that will fit together to make a square.



Homework Problem 3: Visualisation

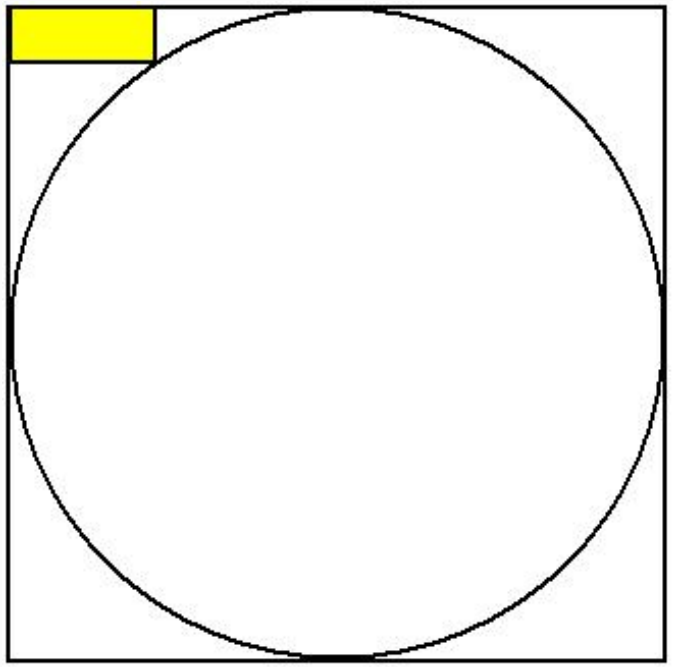
This problem can be solved by inspection (looking at the figure). Two lines can be drawn connecting vertices in the shape as shown in the figure. The three resulting pieces of the shape can then be arranged to form a square as illustrated in the figure below.



Homework Problem 4: Circle inscribed in a square

Problem Statement:

A circle is inscribed in a square, with a rectangle drawn from a corner of the square to a point on the circle, as shown. If this rectangle has side lengths of 6 **cm** and 12 **cm**, what is the radius of the circle?



Homework Problem 4: Circle inscribed in a square - Argument

Every point on a circle has the same distance from the origin. This implies that the rectangle's corner touching the circle must be r centimetres away from the origin, where r is the radius. We can then draw a right-angled triangle with sides $r - y$ and $r - x$. Now, we apply the Pythagorean theorem to this triangle and solve for r:

$$(r - x)^2 + (r - y)^2 = r^2$$

Expanding both sides:

$$r^2 - 2r(x + y) + x^2 + y^2 = 0$$

This is a quadratic equation in r. Applying the quadratic formula:

$$r = \frac{2(x + y) \pm \sqrt{4(x + y)^2 - 4(x^2 + y^2)}}{2}$$

Simplifying further:

$$r = (x + y) \pm \sqrt{2xy}$$

Homework Problem 4: Circle inscribed in a square - Argument (Continued)

Now substituting the given values:

$$r = (6 + 12) \pm \sqrt{2(6)(12)}$$

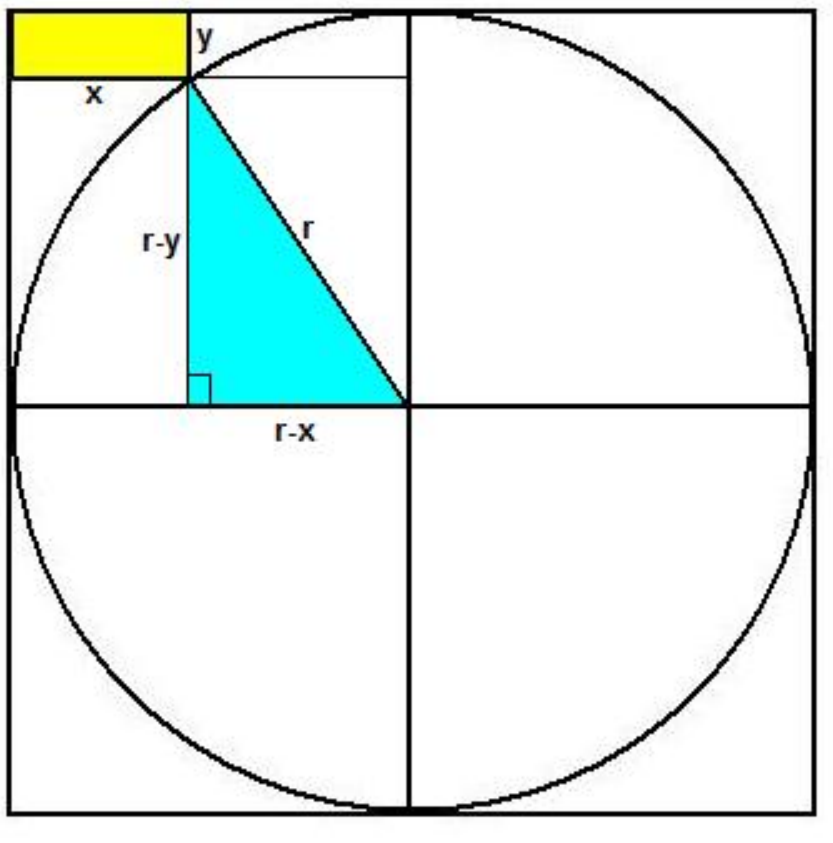
$$r = 18 \pm 12$$

Thus,

$$r = 30 \quad \text{or} \quad r = 6$$

We discard the solution $r = 6$ since this doesn't match the geometry in the question. So, the radius of the circle is 30 **cm**.

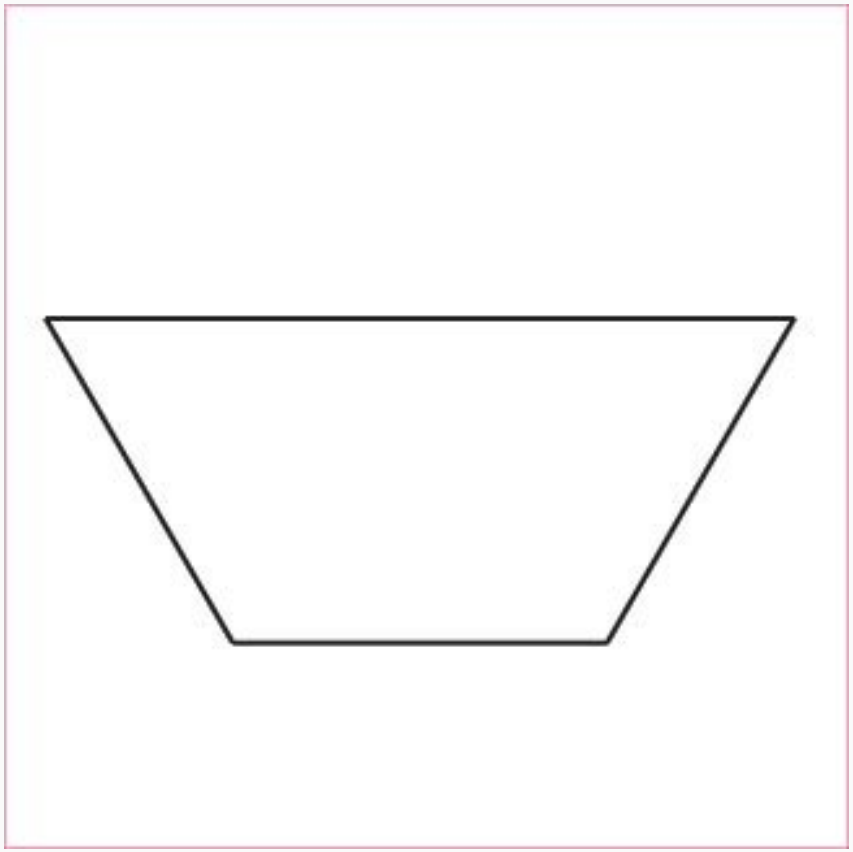
Homework Problem 4: Circle inscribed in a square - Visualisation



Recap

1. Geometry Problem with Dirichlet's Principle

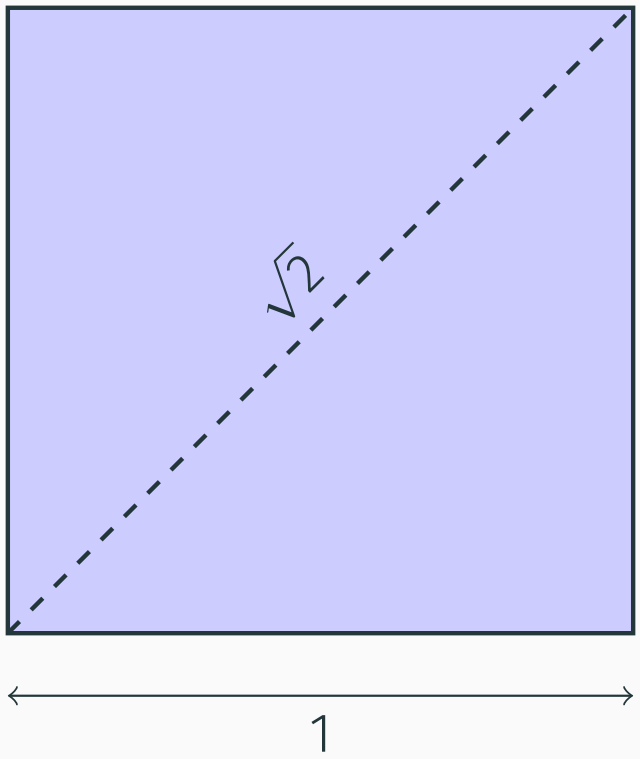
Prove that in any group of 49 points placed inside a regular hexagon of side length 1, at least two points must be at most $\frac{1}{4}$ units apart.



2. Proof by Contradiction - Irrational Numbers

Problem Statement:

Prove that $\sqrt{2}$ is an irrational number.



Mathematical Logic

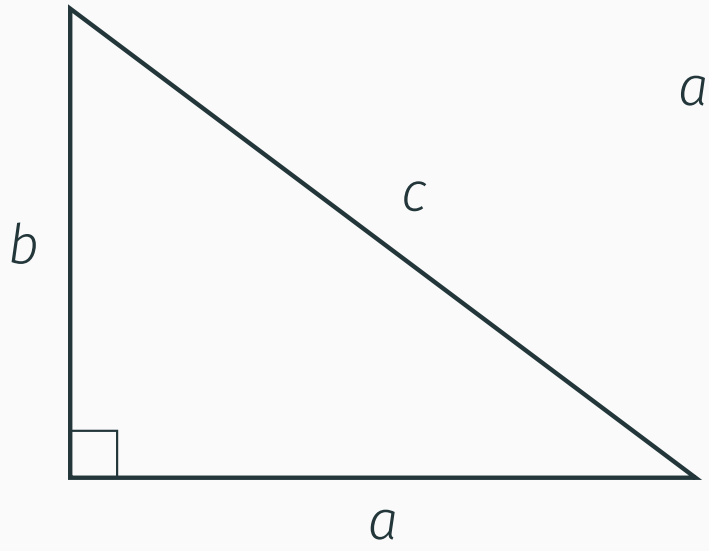
Task 1 - Probability

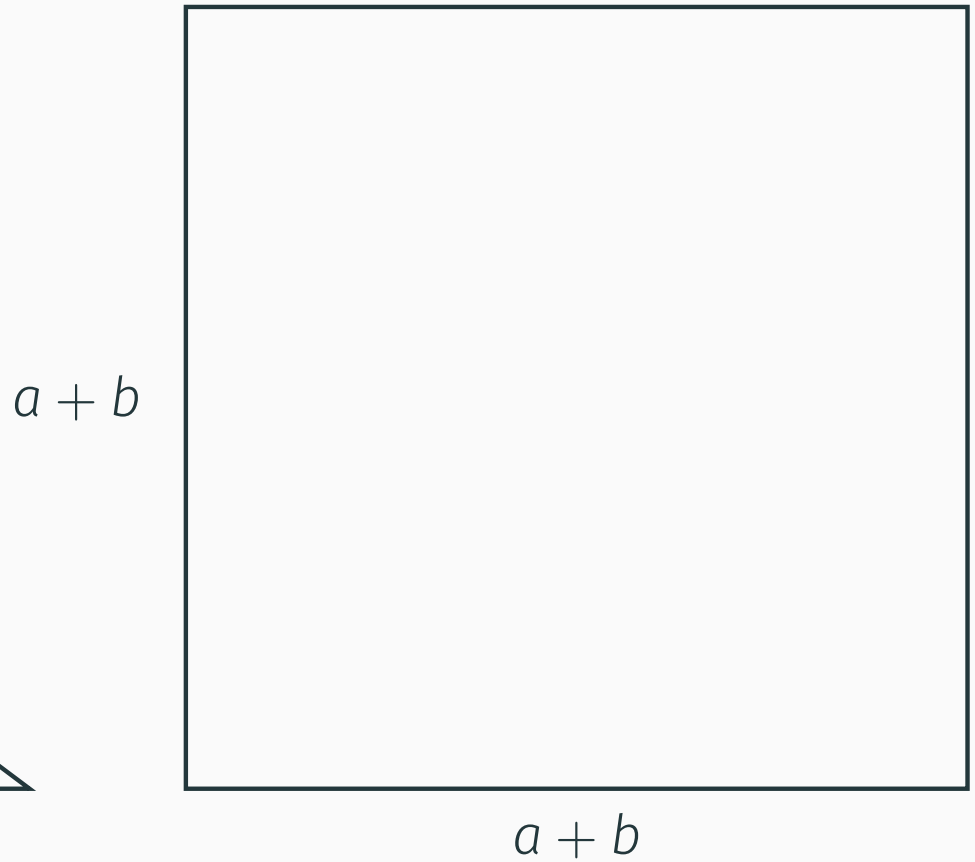
Problem Statement:

An arrow has a 1/4 chance of hitting its target. If four arrows are shot at one target, what's the chance that the target will be hit?

Task 2 - Proving Pythagoras' Theorem!

Problem Statement: Use the provided figures to make a visual proof of Pythagoras' theorem, $a^2 + b^2 = c^2$





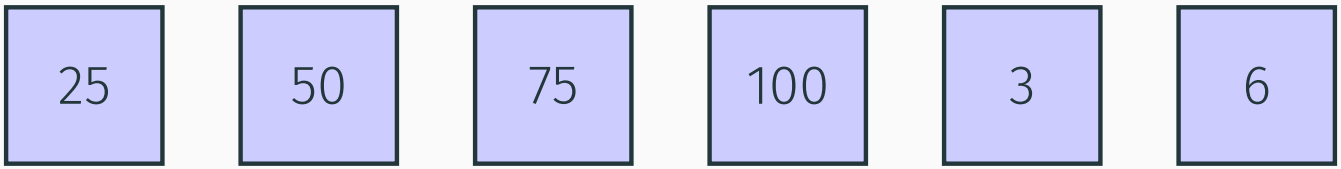
Task 3 - Countdown Numbers Game

Problem Statement

- You are given the numbers: **25, 50, 75, 100, 3, 6**.
- You can use each number **at most once**.
- You can use the operations: **addition (+), subtraction (-), multiplication (×), and division (÷)**.
- Your goal is to reach the **target number: 2,375**.

Question

- How can you combine the numbers and operations to reach the target?

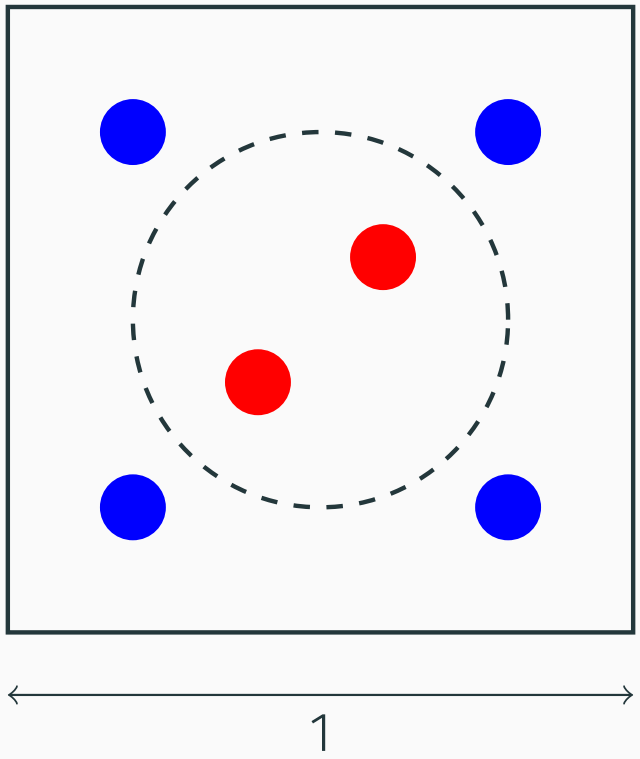


Dirichlet's Principle

Task 4 - Bounding geometry

Problem Statement:

Given 17 points that are placed randomly inside a square of side length 1, prove that there is a circle of radius $\frac{\sqrt{2}}{2}$ that must contain at least 5 points.

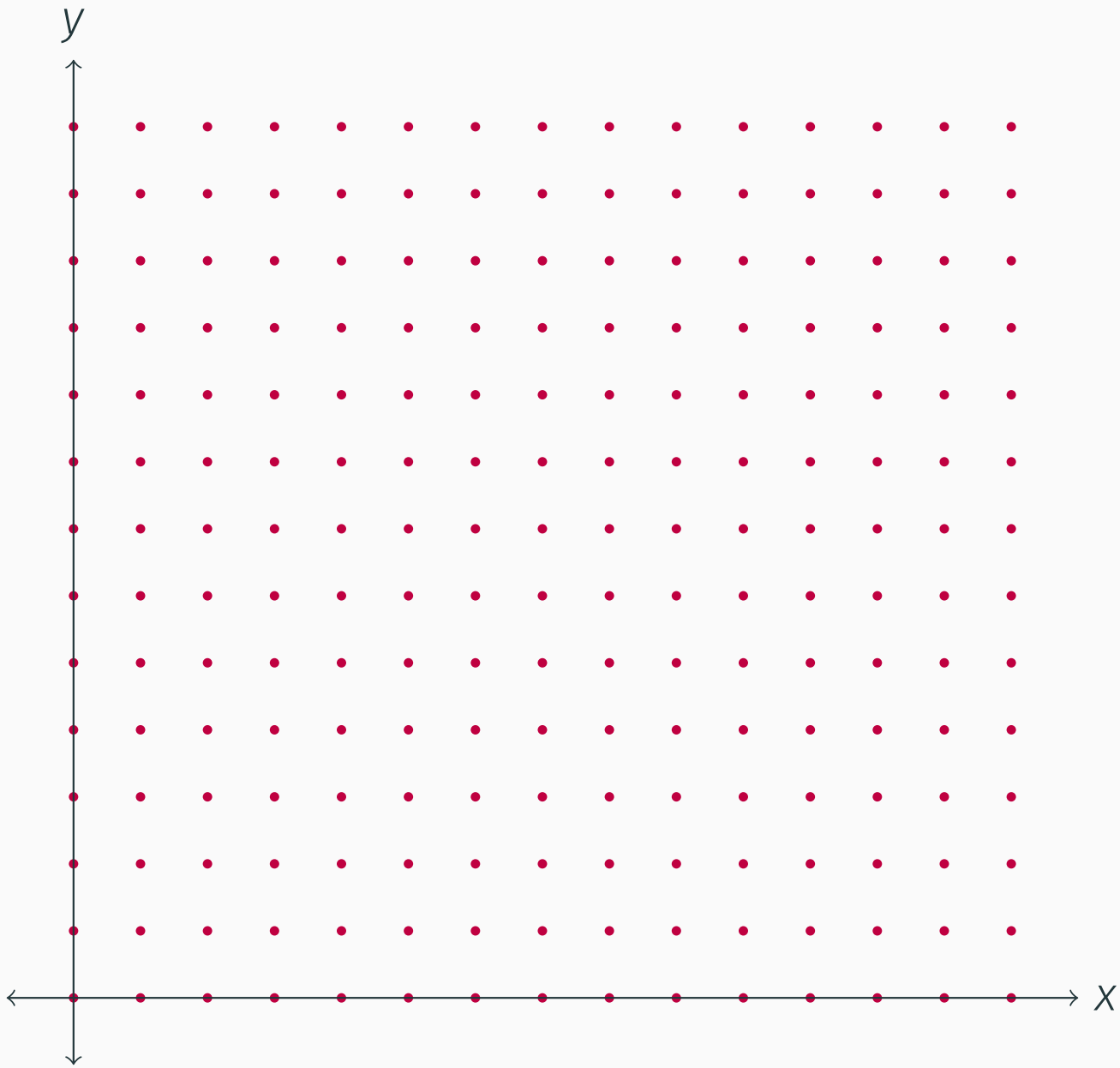


Task 5 - Grid division

Problem Statement:

You are given a rectangular grid of points with dimensions $m \times n$. The goal is to partition this grid into smaller subgrids by dividing it along both dimensions (rows and columns). Determine the most even way to divide the grid such that the sizes of the subgrids are as equal as possible.

Task 5 - Grid division (Visualisation)



Conclusion

Questions?