Mathematical Logic & Dirichlet's Principle

Session 4

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Albion Online Intellectual Club

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Homework Solutions

Homework Problem 1: Number Theory - Palindromic Numbers

Problem Statement: Every four-digit numerical palindrome (e.g., 2772) is a multiple of 11. What is the reason for this fact?

Any such number can be written in the form abba, where a and b are integers, with (a) ranging from 1 to 9 and (b) from 0 to 9. In the Decimal Numeral System we note that,

$$abba = 1000a + 100b + 10b + a = 1001a + 110b = 11(91a + 10b).$$

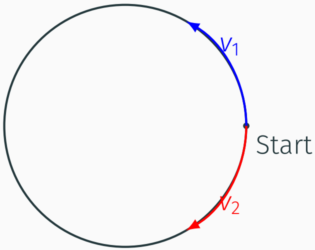
So we can see that 11 is always a factor of a numerical palindrome.

Homework Problem 2: Ratios - Athletes on a track

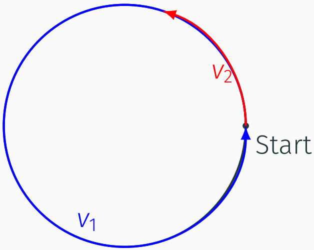
Problem Statement: Two runners start from the same point on a circular track and run at different constant speeds. If they run in opposite directions on the track, they meet after a minute. If they run in the same direction, they meet after an hour. What is the ratio of their speeds?

Homework Problem 2 - Visualisation

Opposite way: Meet after 1 min



Same way: Meet after 60 min



Homework Problem 2: Solution

Let the length of the track be L metres. Assume the speeds of the two runners are v_1 and v_2 metres per second.

- When they run in opposite directions, their relative speed is $v_1 + v_2$. Since they meet after 1 minute (60 seconds):

$$(v_1 + v_2) \times 60 = L$$

- When they run in the same direction, their relative speed is $|v_1 - v_2|$. Since they meet after 1 hour (3600 seconds):

$$|v_1 - v_2| \times 3600 = L$$

Homework Problem 2: Solution (Continued)

From the first equation:

$$v_1 + v_2 = \frac{L}{60}$$

From the second equation:

$$|v_1 - v_2| = \frac{L}{3600}$$

Dividing the two equations:

$$\frac{v_1 + v_2}{|v_1 - v_2|} = \frac{L/60}{L/3600} = 60$$

Let $v_1 = kv_2$ (meaning that person 1 is 'k' times faster than person 2). Substituting this into the two equations:

$$kv_2 + v_2 = 60(kv_2 - v_2)$$

Homework Problem 2: Solution (Continued)

Simplifying:

$$(k+1)v_2 = 60(k-1)v_2$$

Dividing both sides by v_2:

$$k+1 = 60(k-1)$$

Expanding:

$$k+1 = 60k-60$$

$$61 = 59k$$

$$k = \frac{61}{59}$$

The ratio of their speeds is $\frac{61}{59}$.

Homework Problem 3: Number Theory - Prime Factorisation

Problem Statement: Rewrite 1,000,000 as the product of two positive

integers, where neither number contains the digit zero.

First, consider the prime factorisation of 1,000,000:

$$1,000,000 = 10^6 = (2 \times 5)^6 = 2^6 \times 5^6$$

To avoid digits containing zeroes, we can distribute the factors as follows:

$$1,000,000 = 64 \times 15,625$$

Homework Problem 3 - Continued

Here,

$$64 = 2^6 \quad \text{and} \quad 15,625 = 5^6$$

Both 64 and 15,625 contain no zero digits, satisfying the conditions of the problem.

Homework Problem 4: Solution

Problem Statement:

Show that in any group of 29 people, there must be at least 5 individuals who share the same day of the week as their birthday.

There are 7 possible days of the week (Sunday to Saturday) on which a person can be born. By Dirichlet's Principle, if more than 7 items are placed into 7 containers, then at least one container must contain more than one item.

In this problem:

- The 7 days of the week act as the *containers*.
- The 29 people are the *items*.

Homework Problem 4 - Continued

Dividing 29 by 7 gives:

$$\frac{29}{7} \approx 4.14$$

Since the number of people must be an integer, the maximum number of people assigned to any one day (assuming an even spread) would be 4. However, this only accounts for $4 \times 7 = 28$ people.

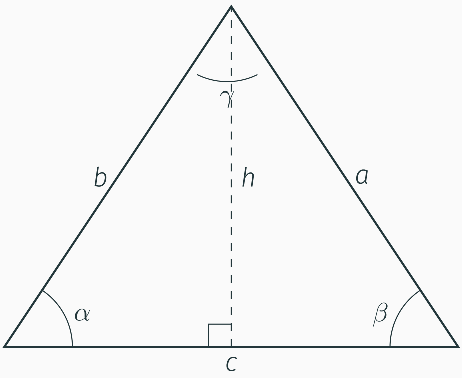
Since there are 29 people, one more person must be added to one of those groups, meaning that at least one day must have 5 people.

So in conclusion, in any group of 29 people, there must be at least 5 individuals who share the same birthday of the week.

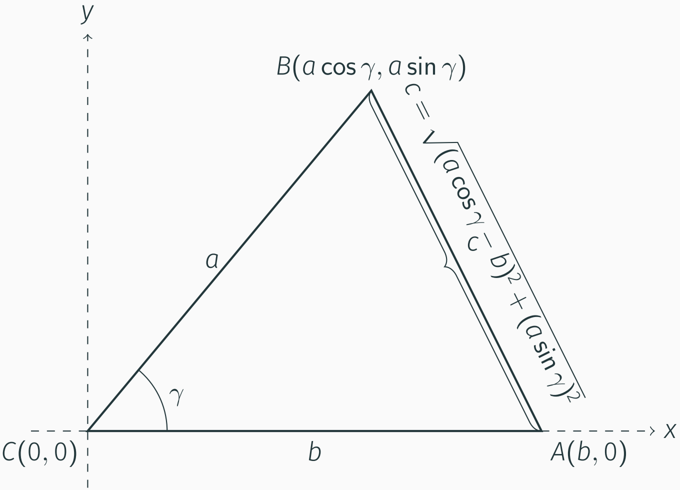
Recap

1. Proving the cosine rule

Prove that $c^2 = a^2 + b^2 - 2ab\cos\gamma$

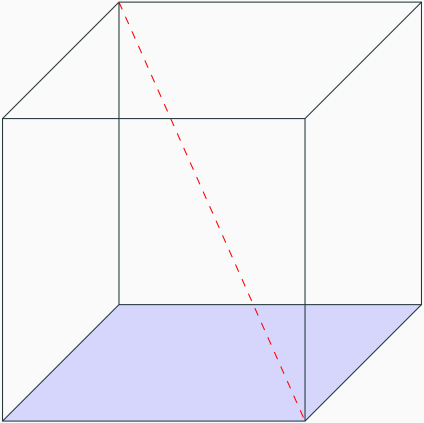


Cosine Rule Visualisation



Problem Statement:

Prove that $\sqrt{3}$ is an irrational number.



Mathematical Logic

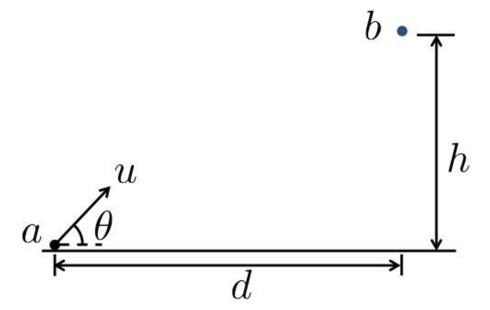
Task 1 - Probability

Problem Statement:

Suppose there is a test for a disease that occurs randomly in every 10,000 people. The test correctly identifies the disease in individuals who are actually infected with a probability of 0.95. This means that the test fails to identify the disease with probability 0.05 in people who have the disease. However, suppose the test also incorrectly identifies the disease with probability 0.001.

If a person tests positive, what is the proabability that they actually have the disease? This is called the **Positive Predictive Value** of the test.

Problem Statement: Particle a is thrown with initial velocity u at an angle θ above the horizontal plane. At the same time, particle b is released from height h above a point which is d away from the initial position of a . Assuming that the air resistance is negligible, the ground surface is perfectly flat and a is capable of reaching b , find the expression or value for θ required for a to collide with b in the sky.



Task 3 - Countdown Numbers Game (Extension)

Problem Statement

- You are given the numbers: **25, 50, 75, 100, 3, 6**.
- You can use each number **at most once**.
- You can use the operations: **addition (+), subtraction (-), multiplication (×), and division (÷)**.
- Your goal is to reach the **target number: 5,614**.

Question

- How can you combine the numbers and operations to reach the target? How many distinct numbers can be made using the 6 numbers listed below?

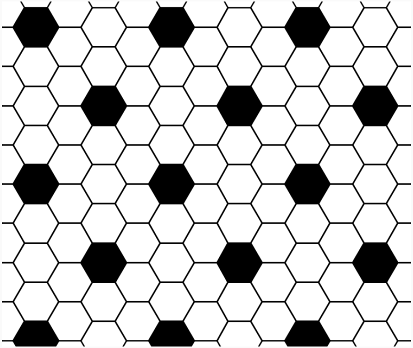
25	50	75	100	3	6

Dirichlet's Principle

Task 4 - Tessellation

Problem Statement:

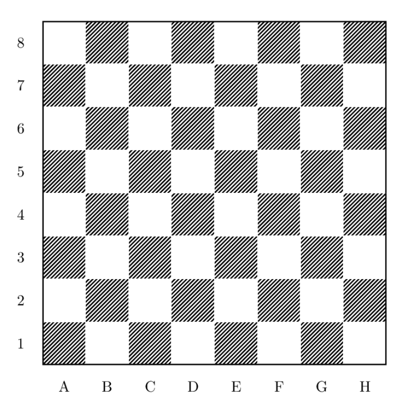
The figure shows part of a tiling, which extends indefinitely in every direction across the whole plane. Each tile is a regular hexagon. Some of the tiles are white, the others are black. What fraction of the plane is black?



Task 5 - Chess Pieces

Problem Statement:

Show that in a 8×8 chess board, it is impossible to place 9 rooks so that they all don't threaten each other. Let's try this problem with other pieces too!



Conclusion

Questions?